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MATHEMATIK

Spektraldarstellung der Kerne. Eine Verallgemeinerung der Sätze von Källén-Lehmann und Herglotz-Bochner u. a.

von

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Einführung

Die klassischen Sätze von Herglotz und Bochner wurden von den Mathematikern meistens in zwei Richtungen verallgemeinert: nach ihrer Unifikation von A. Weil und D. Rajkow (Ersetzung der Gruppe der ganzen Zahlen b. z. w. der reellen Achse durch eine beliebige abelsche lokalkompakte Gruppe) ersetzt man erstens die Exponentialfunktionen durch Charaktere einer lokalkompakten Gruppe. Die Verallgemeinerung des regulären Falles (Bochner'scher Satz) bestand zweitens in der Ersetzung der e^{ix} (Eigenfunktionen des einfachsten Differentialoperators d/idx) durch Eigenfunktionen allemegeinerer Differentialoperatoren.

In zwei vorhergehenden Noten [3], [4] hat der Verfasser eine Verallgemeinerung des regulären Falles (Satz von Bochner) verfolgt: es werden allgemeine positivdefinite Kerne — im Sinne von L. Schwartz — nach gewissen elementaren Kernen entwickelt.

In letzter Zeit wurden von Physikern (vgl. z. B. [6]) gewisse — im allgemeinen nicht positivdefinite — Distributionen, die s. g. Vakuumserwartungswerte der Quantenfelder nach Eigendistributionen der Wellenoperatoren entwickelt (der s. g. Satz von Källén-Lehmann).

In der vorliegenden Note versucht der Verfasser zu zeigen, wie diese verschiedene Verallgemeinerungen der Sätze von Herglotz und Bochner unter einen Hut zu bringen sind: man gewinnt Spektraldarstellungen ziemlich allgemeiner Kerne nach Eigenkernen der vertauschbaren Operatoren in einem passend gewähltem Hilbertschen Raume. Es erweist sich z. B. dass in dem diskreten Fall (Verallgemeinerung des Herglotzschen Satzes) schon die Algebra der Verschiebungsoperatoren das Nötige leistet. In diesem Falle bekommt man als Korollar einen — wie es scheint neuen — Beweis des Satzes von Pontriagin: "die Charakterengruppe der diskreten Gruppe ist kompakt".

1. Definitionen und Hilfsmittel

Es sei H ein separabler Hilbertscher Raum mit dem Skalarprodukt (u, v), $u, v, \in H$; und es sei (A_ν) ein vertauschbares System der selbstadjungierten (b. z. w. normalen) Operatoren $A_\nu = A_\nu^*$ (b. z. w. $A_\nu A_\nu^* = A_\nu^* A_\nu$), dann existiert ein direktes Integral der Hilbert'schen Räume $\hat{H}(\lambda)$

$$\hat{H} = \int_\Lambda \oplus \hat{H}(\lambda) d\mu(\lambda),$$

wo μ ein positives Mass auf der kompakten separablen Menge Λ ist und eine (unitäre) Abbildung $F: H \to \hat{H}$, die das System (A_ν) gleichzeitig diagonalisiert d. h:

$$\begin{gathered} H \ni u \to \hat{u}(\cdot) = Fu \in \hat{H}, \quad \text{wo} \quad \hat{u}(\lambda) \in \hat{H}(\lambda), \ \lambda \in \Lambda \\ \widehat{(A_\nu u)}(\lambda) = \hat{A}_\nu(\lambda)\hat{u}(\lambda) \end{gathered} \tag{1}$$

wo $\hat{A}_\nu(\cdot)$ skalare μ-messbare und μ-fast überall endliche Funktionen sind. Die Abbildung F ist unitär d. h:

$$(u, v) = \int_\Lambda (\hat{u}(\lambda), \hat{v}(\lambda))_\lambda d\mu(\lambda); \quad (\cdot, \cdot)_\lambda \tag{2}$$

bedeutet das skalare Produkt im Hilbert'schen Raume $\hat{H}(\lambda)$. Falls A_ν unitär ist:

$$A_\nu^* = A_\nu^{-1}, \quad \text{dann ist} \quad |\hat{A}_\nu(\lambda)| = 1 \quad \text{für} \quad \text{f. a. } \lambda \in \Lambda. \tag{3}$$

Im Falle der endlich vielen Erzeugenden $A_1, \ldots, A_n$ ist es oft günstig mit der s. g. kanonischen Diagonalisation zu arbeiten:

$$\begin{gathered} \widehat{(A_\nu u)}(\lambda) = \lambda_\nu \hat{u}(\lambda), \quad \lambda_\nu \in \operatorname{sp} A_\nu \ (\text{Spektrum von } A_\nu) \\ \lambda = (\lambda_1, \ldots, \lambda_N) \in \Lambda = \prod_{\nu=1}^{N} \operatorname{sp} A_\nu \subset \mathcal{C}^N. \end{gathered} \tag{4}$$

Wenn alle A_ν selbstadjungiert sind, dann kann man Λ als lokalkompakte Untermenge des N-dimensionalen Euklidischen Raumes nehmen.

Falls B ein beschränkter mit allen A_ν vertauschbarer Operator ist, dann ist B zerlegbar d. h.

$$\widehat{(Bu)}(\lambda) = \hat{B}(\lambda)\hat{u}(\lambda) \quad \lambda \in \Lambda, \tag{5}$$

wo $\hat{B}(\lambda): \hat{H}(\lambda) \to \hat{H}(\lambda)$ ein beschränkter Operator im Raume $\hat{H}(\lambda)$ ist.

Nach dieser Vorbereitung können wir zur eigentlichen Untersuchung vorschreiten.

2. Der (reguläre) Fall der differenzierbaren Mannigfaltigkeit

Es sei Ω eine differenzierbare Mannigfaltigkeit und $\mathcal{D}' = \mathcal{D}'(\Omega)$ der Raum der Schwartz'schen Distributionen auf Ω d. h der Raum der stetigen Funktionale auf dem Raume $\mathcal{D}$ der beliebig oft differenzierbaren Funktionen mit kompakten Trägern in Ω.

Es sei (A_ν) ein System der auf $\mathcal{D}$ vertauschbaren Operatoren, die $\mathcal{D}$ stetig in sich abbilden. Es sei weiter $K(\varphi, \psi) = (\varphi, \psi)$ eine stetige positivdefinite hermitische Form auf $\mathcal{D}$: d. h. ein Kern (im Sinne von L. Schwartz) auf Ω. Es sei schliesslich H der durch die Vervollständigung des prehilbert'schen Raumes $\mathcal{D}$ entstandene Hilbertsche Raum. Die Operatoren A_ν induzieren in H die Operatoren $\dot{A}_\nu$, welche vertauschbare selbstadjungierte (normale) Fortsetzung $\bar{\dot{A}}_\nu$ besitzen sollen. Im Folgenden werden diese drei Arten von Operatoren $A_\nu, \dot{A}_\nu, \bar{\dot{A}}_\nu$ mit A_ν bezeichnet. Jetzt können wir unseren ersten Satz beweisen:

Satz 1. *Es sei B ein Kern auf Ω, der einen Operator B in H induziert (d. h. die bilineare Form $B(\varphi, \psi)$ ist separatstetig auf $\mathcal{D} \times \mathcal{D}$ in der H-Topologie). Es sei (A_ν) ein obengenanntes vertauschbares System der Operatoren in H. Es werden folgende Voraussetzungen gemacht:*

1° *die Abbildungen $A_\nu: \mathcal{D} \to \mathcal{D}$ sind stetig;*

2° *der Operator B ist vertauschbar mit allen A_ν auf der Menge $\mathcal{D}$: $BA_\nu\varphi = A_\nu B\varphi$, $\varphi \in \mathcal{D}$;*

3° *die Transformation B ist stetig auf $\mathcal{D}$. (d. h wegen 2°: $B(A_\nu\varphi, \psi) \equiv B(\varphi, A_\nu\psi)$, $\varphi, \psi \in \mathcal{D}$). (Statt 3° kann man die folgende — nicht äquivalente — Voraussetzung machen: 3°′ B ist beschränkt auf H).*

Dann hat der Kern $B(\cdot, \cdot)$ die folgende Spektraldarstellung:

$$B(x, y) = \int_\Lambda B_\lambda(x, y)\, d\mu(\lambda) \tag{6}$$

oder genauer

$$\langle \varphi, B\psi \rangle = (\varphi, B\varphi) = \int_\Lambda \langle \varphi, B_\lambda \psi \rangle\, d\mu(\lambda) \tag{7}$$

wo die Kerne $B_\lambda \in \mathcal{D}' \otimes \mathcal{D}'$ Eigenkerne der Operatoren A_ν sind d. h.:

$$A_\nu^x B_\lambda(x, y) = \hat{A}_\nu(\lambda) B_\lambda(x, y) = A_\nu^y \overline{B_\lambda(x, y)}; \tag{8}$$

genauer es gilt

$$\langle \varphi, B_\lambda A_\nu \psi \rangle = \langle A_\nu \varphi, B_\lambda \psi \rangle = \overline{\langle A_\nu \psi^{tr}, B_\lambda \varphi \rangle} = \hat{A}_\nu(\lambda) \langle \varphi, B_\lambda \psi \rangle\,. \tag{9}$$

Falls der Kern B positivdefinit ist, sind auch fast alle Eigenkerne B_λ positivdefinit identisch für $\varphi, \psi \in \mathcal{D}$ und f. a. $\lambda \in \Lambda$.

Beweis. Aus dem Gårding'schen Spektralsatze (vgl. [1] u. [3]) wissen wir, dass die Abbildung

$$\mathcal{D} \ni \psi \to \hat{\psi}(\lambda) \in \hat{H}(\lambda) \tag{10}$$

stetig ist, daraus und aus 3° (3°') folgt es, dass auch die Abbildung

$$\mathcal{D} \ni \varphi \to \widehat{B\varphi}(\lambda) \in \hat{H}(\lambda) \tag{11}$$

stetig ist, also ist die bilineare Form

$$\{\varphi, \psi\} \to (\hat{\varphi}(\lambda), (\widehat{B\psi})(\lambda))_\lambda \quad \text{separatstetig auf } \mathcal{D} \times \mathcal{D}.$$

Es existiert also (für fast alle $\lambda \in \Lambda$) ein solcher Kern $B_\lambda \in \mathcal{D}'(\Omega \times \Omega)$, dass

$$(\hat{\varphi}, (\widehat{B\psi})(\lambda))_\lambda \equiv \langle \varphi, B_\lambda \psi \rangle \tag{12}$$

identisch für $\varphi, \psi \in \mathcal{D}$ gilt. Aus der Vertauschbarkeit: $BA_\nu = A_\nu B$, der Selbstadjungiertheit von A_ν^* und (12) haben wir

$$(\hat{\varphi}(\lambda), (\widehat{BA_\nu\psi})(\lambda))_\lambda \equiv (\hat{\varphi}((\lambda), (\widehat{A_\nu B\psi})(\lambda))_\lambda = (\widehat{A_\nu\varphi}(\lambda), \widehat{B\psi}(\lambda))_\lambda$$

$$\equiv \hat{A}_\nu(\lambda)(\hat{\varphi}(\lambda), \widehat{B\psi}(\lambda))_\lambda, \qquad \varphi, \psi \in \mathcal{D}$$

wegen (12) kann man diese Identitäten eben in der Form (9) schreiben.

Eine Verallgemeinerung.

Man könnte natürlich dieselben Überlegungen anstellen für die bilineare Formen auf den endlichdimensionalen Vektorfeldern $\vec{\varphi}(\cdot)$

$$\Omega \ni x \to \vec{\varphi}(x) \in V,$$

wo V ein endlichdimensionaler Vektorraum ist.

Solche "vektorielle" Kerne werden z. B. in der Quantenmechanik der Wellenfelder benutzt (vgl. den nächsten §).

Korollar. *Falls der Kern B mit dem Kern K identisch ist, bekommen wir den in* [3] *bewiesenen Satz (in diesem Falle ist der durch den Kern induzierte Operator $K = I$)*

$$K(x, y) = \int_\Lambda K_\lambda(x, y)\, d\mu(\lambda).$$

In diesem Falle sind $K_\lambda(.,.)$ positivdefinit.

3. Anwendungen von Satz 1

Als Anwendungen bekommt man eine Verallgemeinerung der von dem Verfasser in [4] bewiesenen Sätze über die Entwicklung der Kerne (auch nichtpositivdefiniten) auf den homogenen Räumen insbesondere auf Lie'schen Gruppen nach Eigenkernen der verschiebungsinvarianten Differentialoperatoren. Als Beispiel führen wir eine weitgehende Verallgemeinerung des Spektralsatzes von Källén und Lehmann [7] zu.

Es sei $\Omega = (\Omega, G)$ ein (homogener) Raum auf dem eine separable (zusammenhängende) Lie'sche Transformationsgruppe G transitiv wirkt: $\Omega \ni x \to g^{-1}x \in \Omega$, $g \in G$. Den positivdefiniten Kern setzen wir jetzt als invariant voraus, d. h.:

$$K(\varphi, \psi) = (\varphi, \psi) = (L_g\varphi, L_g\psi)$$

identisch für

$$g \in G;\ \varphi, \psi \in \mathfrak{D} = \mathfrak{D}(\Omega) \quad \text{wo} \quad (L_g\varphi)(x) \overset{\text{df}}{=} \varphi(g^{-1}x)\,.$$

Es sei

$$\mathfrak{G} \overset{\text{df}}{=} \left\{u \in H : u = \int_G a(g) L_g v dg\,,\ v \in H\right\}$$

der Gårding'sche Unterraum von H.

Es seien (A_ν) die vertauschbaren verschiebungsinvarianten Differentialoperatoren in H (wegen ihrer Konstruktion vgl [4]). Um eine kanonische Spektraldarstellung zu erhalten, setzen wir voraus, dass das System (A_ν) endlich viele erzeugenden $A_1, \ldots, A_N$ besitzt (vgl § 1). Wir haben jetzt als Korollar zu Satz 1 den folgenden

Satz 2. *Es sei B ein invarianter Kern auf dem homogenem Raum $\Omega = (\Omega, G)$, der einen abgeschlossenen Operator B in H induziert:*

$$B : D \to D(A_\varkappa) \qquad \varkappa = 1, \ldots, k\,.$$

Dann besitzt der Kern B eine Spektraldarstellung

$$B(x, y) = \int_{\Pi \mathfrak{sp} A_\nu} B_\lambda(x, y) d\mu(\lambda)\,,$$

wo die Kerne (invariante) Eigenkerne der verschiebungsinvarianter Differentialoperatoren (der auch s. g. Laplace-Operatoren) auf dem homogenen Raume Ω sind.

Beweis. Wie wir wissen, braucht man nur zu zeigen, dass der induzierte Operator B mit den Operatoren (A_ν) auf der Menge D vertauschbar ist. Da die A_ν Polynome der Lie'schen Differentialoperatoren $X_1, \ldots, X_k$ auf Ω sind, genügt es also die Vertauschbarkeit von B mit $X_1, \ldots, X_k$ $r = 1, \ldots, k$ zu beweisen.

Es sei

$$X_\varkappa \varphi \overset{\text{df}}{=} \lim_{t \to 0} \left(\frac{L_{a_\varkappa} - I}{t}\right)\varphi\,.$$

Da $B(\varphi, \psi) = (B\varphi, \psi)$, haben wir wegen der Invarianz des Kernes B

$$(B\varphi, \psi) = B(\varphi, \psi) = B(L_g\varphi, L_g\psi) = (BL_g\varphi, L_g\psi)$$

identisch für $\varphi, \psi \in D$; $g \in G$ also $B = L_g^* B L_g = L_g^{-1} B L_g$, d. h.,

$$L_g B = B L_g\,.$$

Aber

$$X_\varkappa \varphi \overset{\text{df}}{=} \lim t^{-1}(L_{a_\varkappa(t)} - I)\varphi .$$

Wegen der Vertauschbarkeit der Operatoren B und $L_{a_\varkappa(t)}$ haben wir also

$$Bt^{-1}(L_{a_\varkappa(t)} - I)\varphi = t^{-1}(L_{a_k(t)} - I)B\varphi \underset{t\to 0}{\to} X_k B\varphi .$$

Wegen der Abgeschlossenheit des Operators B konvergiert aber die linke Seite obiger Gleichung gegen $BX_k\varphi$.

Wir haben also

$$BX_k\varphi = X_k B\varphi \quad \text{identisch für} \quad \varphi \in \mathcal{D}.$$

Es gilt ein analoger Satz für vektorielle Kerne.

Beispiel. Im Falle des homogenen Raumes $M_4 = (M_4, L)$, wo M_4 – der Lorentz-Minkowski Raum und L die Orthochrone Lorentzgruppe ist, bekommen wir die Spektraldarstellungen

$$b(x-y) = B(x,y) = \int_{-\infty}^{\infty} B_\lambda(x,y)\,d\mu(\lambda) = \int_{-\infty}^{\infty} b_\lambda(x-y)\,d\mu(\lambda) ,$$

wo $b_\lambda(\cdot)$ die Eigendistribution des Wellenoperators ist

$$(\Box - \lambda) b_\lambda(\cdot) = 0 .$$

Im Falle der vektoriellen Kerne (Fall der Vektor, Spinor, Tensor-Felder) sind die $b_\lambda^\varkappa$ Eigenlösungen gewisser Lorentzinvarianter Systeme der Differentialoperatoren (z. B. des Dirac-Operators).

4. K-Carleman'sche verschiebungsinvariante Operatoren

Der Behandlung des Falles der diskreten Gruppe setzen wir eine allgemeine Betrachtung über Carleman'sche Operatoren auf einem homogenem Raume voraus (vgl. [5]).

Es sei jetzt $\Omega = (\Omega, G)$ ein lokalkompakter Raum auf dem eine Transformationsgruppe G transitiv wirkt. Auf der Menge $\mathcal{D}^0 = \mathcal{D}^0(\Omega)$ der stetigen Funktionen mit kompakten Trägern in Ω mit der üblichen Topologie ist wieder eine verschiebungsinvariante positivdefinite Form $K(\varphi,\psi) = (\varphi,\psi) = K(L_g\varphi, L_g\psi)$ erklärt.

Es sei $a(\cdot,\cdot)$ eine hermitisch-symmetrische messbare Funktion auf $\Omega\times\Omega$. Wir geben jetzt eine Verallgemeinerung des Carleman'schen Operators.

Definition. *Der Integraloperator A im Raume H*

$$(Au)(x) \overset{\text{df}}{=} \int_\Omega a(x,y)u(y)\,d\sigma(y) \quad \textit{für} \quad u \in D(A) \supset \mathcal{D}^0$$

heisst K-Carleman'sch falls:

$$a(x, \cdot) \in H\,, \quad \text{d. h.} \quad a^2(x) \overset{\text{df}}{=} \|a(x, \cdot)\|^2 < \infty\,.$$

Es gilt das folgende

LEMMA: *Falls der K-Carleman'sche Operator verschiebungsinvariant ist d. h. falls*

$$L_g A\varphi = AL_g\varphi \qquad \varphi \in \mathcal{D}^0\,.$$

Dann ist

1° $a(gx, \cdot) = L_g a(x, \cdot)$, $g \in G$.

2° $c^2(x) = c^2(gx) = \text{const.}$ f. ü. auf Ω

Beweis. Man hat identisch für $\varphi \in \mathcal{D}^0$

$$\int a(gx, y)\varphi(y)d\sigma(y) = L_{g^{-1}}A\varphi(x) = AL_{g^{-1}}\varphi(x)$$

$$= \int a(x, y)\varphi(gy)\sigma d(y) = \int a(x, g^{-1}y)\varphi(y)d\sigma(y)$$

(Invarianz des Masses σ), also

$$a(gx, \cdot) = L_g a(x, \cdot);$$

also

$$c^2(gx) = \|a(gx, \cdot)\|^2 = \|L_g a(x, \cdot)\|^2 = \|a(x, \cdot)\| = c^2(\ )\,.$$

Wenn man jetzt (genau) die Überlegungen in [1] und [5] mutatis muntandis wiederholt, bekommt man den folgenden

SATZ 3. *Die Eigendistributionen $e(\cdot, \lambda)$ eines K-Carleman'schen verschiebungsinvarianten Operators A auf einem homogenen Raume $\Omega = (\Omega, G)$ sind (für μ — f. a. λ) wesentlich beschränkte Funktionen auf Ω. Die von dem Operator A induzierte Fouriertransformation*

$$\mathcal{D}^0 \ni \varphi \to \hat{\varphi}(\lambda) \in \hat{H}(\lambda)$$

ist stetig, sie ist gegeben durch die folgende Formel

$$\hat{\varphi}(\lambda) = \int_{\Omega} \overline{e(x, \lambda)}\varphi(x)d\sigma(x)$$

wobei

$$\hat{A}(\lambda)e(x, \lambda) \overset{\text{df}}{=} (Fa(x, \cdot))(\lambda)\,.$$

Es sei insbesonders $\Omega = G$ — eine diskrete (lokalkompakte) Gruppe. Das Mass σ ist normiert $\sigma(\{g\}) = 1$, $g \in G$.

Jetzt sind Verschiebungsoperatoren Integraloperatoren mit dem Kronecker-Dirac'schen Kern: $\delta(g^{-1}x, a) = \delta_{g^{-1}x,a}$

$$(*) \qquad (L_g\varphi)(x) = \varphi(g^{-1}x) = \sum_{a \in G} \delta(g^{-1}x, a)\varphi(a)\,.$$

Der Verschiebungsinvariante Kern $K(\cdot,\cdot)$ hat jetzt die folgende Form: $K(x,y) = k(x^{-1}y)$, wo $k(\cdot)$ eine (stetige) Funktion auf G ist:

$$K(\varphi,\psi) = k(\varphi,\psi) = \int k(x^{-1}y)\varphi(x)\overline{\psi(y)}\,d\sigma(x)\,d\sigma(y) = \sum_{x,y\in G} k(x^{-1}y)\varphi(x)\overline{\psi(y)}\,.$$

Aus (*) folgt augenblicklich das

Lemma. *Die Verschiebungsoperatoren auf der diskreten Gruppe sind K-Carleman'sch.*

Beweis.

$$\|\delta(x,\cdot)\|^2 = \sum_{z,y} k(z^{-1}y)\,\delta(x,z)\,\delta(x,y) = k(x^{-1}x) = k(e) = \text{const} < \infty\,.$$

5. Verallgemeinerung des Satzes von Hergoltz-Weil-Rajkow

Im folgenden setzen wir voraus, dass der homogene Raum $\Omega = G$ eine diskrete (lokalkompakte) abelsche Gruppe ist. Wie wir wissen, sind in diesem Falle Verschiebungsoperatoren L_g 1° unitär (im Raume H); 2° vertauschbar; 3° K-Carleman'sch.

Die Überlegungen der vorigen §§ führen jetzt also zum

Satz 4. *Der verschiebungsinvariante Kern $B(g_1,g_2) = b(g_1^{-1}g_2)$ auf der diskreten abelschen Gruppe G genüge der Voraussetzungen des Satzes 1 (order Satzes 2), darum besitzt er die folgende Spektraldarstellung*

$$b(g) = \int_\Lambda b_\lambda(g)\,d\mu(\lambda);$$

*die $b_\lambda(\cdot)$ sind jetzt (für μ-fast alle $\lambda \in \Lambda$) beschränkte Eigenfunktionen der Verschiebungsoperatoren L_g *).*

Wenn wir $B = K$ nehmen, bekommen wir den folgenden

Korollar (Satz von Hergoltz-Weil-Rajkow).

Die positivdefinite Funktion auf der diskreten abelschen Gruppe G besitzt die folgenden Spektraldarstellungen

$$(**) \qquad k(g) = \int_\Lambda k(g)\,d\mu(\lambda)$$

oder wenn wir die Eigenfunktionen normieren:

$$(N) \qquad \chi_\lambda(g) \overset{\text{df}}{=} \frac{k_\lambda(g)}{k_\lambda(e)}; \quad d\mu'(\lambda) \overset{\text{df}}{=} k_\lambda(e)\,d\mu(\lambda)$$

$$k(g) = \int_\Lambda \chi_\lambda(g)\,d\mu'(\lambda)\,.$$

*) Die Existenz der Eigenfunktionen (aber nicht ihre Beschränktheit) der Verschiebungsoperatoren auf diskreten abel'schen Gruppen folgt auch aus einem abstrakten Satze von Gelfand u. Kostiučenko, vgl. [2] S. 210.

Die so normierten Eigenfunktion der Verschiebungsoperatoren sind Charaktere der Gruppe G.

Beweis. Wir können den trivialen Fall $k(g) = 0$ ausschliessen. Zuerst zeigen wir, dass die Normierung (N) erlaubt ist:

Wir wissen, dass alle $k_\lambda(\cdot)$ positivdefinit sind: $k_\lambda(e) \geqslant 0$. Wäre aber $k_\lambda(e) \equiv 0$ für f. a λ, dann hätten wir

$$k_\lambda(g) = L_{g^{-1}} k_\lambda(e) = \hat{L}_{g^{-1}}(\lambda) k_\lambda(e) = 0$$

für fast alle $\lambda \in \Lambda$, also folgt aus $(**)$ $k(g) = 0$ was wir ausgeschlossen haben.

Es bleibt also zu zeigen dass $\chi_\lambda(\cdot)$ (beschränkte) Charaktere der Gruppe G sind: $\chi_\lambda(e) = 1$, daher

$$\hat{L}_g(\lambda) = L_g(\lambda) \cdot 1 = \hat{L}_g(\lambda) \cdot \chi_\lambda(e) = L_g \chi_\lambda(e) = \chi_\lambda(g^{-1});$$

aus dem Spektralsatze (Diagonalisation des Verschiebungsoperators L_g) haben wir also

$$\chi_\lambda(g_1 \cdot g_2) = \hat{L}_{(g_1 \cdot g_2)^{-1}}(\lambda) = \hat{L}_{g_2^{-1}}(\lambda) \hat{L}_{g_2^{-1}}(\lambda) = \chi_\lambda(g_1)\, \chi_\lambda(g_2)\,.$$

Da die Verschiebungsoperatoren unitär sind, haben wir aus (3)

$$|\chi_\lambda(g)| = |\hat{L}_{g^{-1}}(\lambda)| = 1\,.$$

Der Beweis ist also vollständig erbracht.

Wenn wir in der Menge der Charakteren $(\chi_\lambda(\cdot))$ die Topologie der (kompakten) Menge Λ einführen, bekommen wir als weiteres

Korollar (Satz von Pontriagin): *Die Charakterengruppe $\hat{G}$ der diskreten abelschen Gruppe G ist kompakt.*

Bemerkung. Den *klassischen Satz von Herglotz* könnte mann unmittelbar aus der folgenden Bemerkung erhalten:

Auf der Gruppe Z der Ganzen Zahlen besitzen wir im wesentlichen einen Verschiebungsoperator

$$L_1 : (L_1 \varphi)(n) = \varphi(n-1)\,.$$

Die kanonische Gestalt des Spektralsatzes für den unitären Operator liefert also den Herglotz'schen Satz in klassischer Form.

$$k(n) = \int_0^1 e^{2\pi i n \lambda} d\mu'(\lambda),$$

weil die Eigenfunktionen des Verschiebungsoperators L_1 normiert sind.

$$\chi_\lambda(n) = e^{2\pi i \lambda n}\,.$$

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MATHEMATIK

Allgemeine Eigenfunktionsentwicklungen. Spektraldarstellung abstrakter Kerne. Eine Verallgemeinerung der Distributionen auf Lie'schen Gruppen

von

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Einführung

Um dem Spektralsatz für selbstadjungierte Operatoren im (abstrakten) Hilbert'schen Raum eine mehr "klassische" Form zu geben haben Gelfand und Kostiučenko [4] ein Schema entworfen, das als Ausgangspunkt einen nuklearen Fréchet-(B_0-) Raum Φ und den Gelfand'schen Satz über die Differentiation der Funktionale von beschränkter Variation nahm. Diese elegante Methode leidet jedoch an folgenden Nachteilen: die in der Praxis vorkommenden Räumen sind oft nicht metrisierbar: z.B. $\mathcal{D}(\Omega)$, Ω — offenes Bereich; es wurde nicht die Konstruktion des Raumes angegeben und schliesslich werden sehr tiefe Hilfsmittel aus der Differentiationstheorie benutzt.

In wesentlich derselben Richtung bewegt sich die interessante Note von G. I. Kac [7], wo ein abstrakter Hintergrund der Sätze von Typus Gårding-Berezanskij [5], [1] aufgezeigt wird. Der Ausgangspunkt war ein Banach'scher Unterraum Φ von H *). Diese Note gab Impuls den Untersuchungen von C. Fojas [2], [3], der bemerkt hat, dass die Konstruktion von Kac im wesentlichen die Nuklearität der Einbettung (Injektion) $\Phi \to H$ benutzte **).

Im ersten Abschnitt der vorliegenden Note wird gezeigt, wie man mit elementaren Hilfsmitteln (Satz von Fubini und Schwarz'sche Un-

*) Bei diesen Sätzen verlangt man nicht, dass der selbstadjungierte Operator $A_\tau : \Phi \to \Phi$ stetig überführt. Auf diese Weise bekommt man weniger scharfe Aussagen über die Natur der „Eigenelemente" $e_t(\lambda)$.

**) In einem anderen Gedankenkreise bewegt sich die demnächst zu erscheinende C.R.-Note von J. Odhnoff.

gleichung) eine Verallgemeinerung der obengenannten Sätze gewinnen kann. Das im Abschnitt 1. bewiesene Lemma erlaubt uns ein allgemeines Schema (Satz 2) für Spektraldarstellung der abstrakten (auch nichtpositiven) Kerne d. h. der Elemente von $\Phi' \otimes \Phi'$ (Φ — nuklear) zu entwerfen. Dieser Satz umfasst also die in vorhergehender Note [9] von uns bewiesenen Sätze vom Typus Herglotz-Bochner-Weil-Rajkow sowie — Källén-Lehmann.

Um im abstrakten Falle die Existenz des vollständigen Systems der Eigenelemente zu zeigen muss man den im abstrakten Schema postulierten nuklearen Raum Φ konstruieren. Das geschieht im dritten Abschnitt für den Fall eines abzählbares Systems der Operatoren, was also einen in [3] ohne Beweis angesagten Satz (n selbstadjungierte Operatoren) umfasst.

Bekanntlich sind die Charaktere der regulären Darstellung L_g Lie'scher Gruppe G im allgemeinen keine Funktionen sondern Distributionen (eigentlich Elemente von $\mathcal{D}'(G) \otimes \mathcal{D}'(G) = \mathcal{D}'(G \times G)$). Wie in [8] gezeigt wurde, sind in diesem Falle Charaktere einer irreduziblen (oder Faktor-) Darstellung Eigenkerne der verschiebungsinvarianten Differentialoperatoren (Laplace-Operatoren in der Terminologie von Gelfand). Um die Spektralsätze von [8], [9] auch auf den Fall beliebiger unitären Darstellung $T(g)$ der Lie'schen Gruppe G zu übertragen lag es nahe den Begriff der Distribution zu erweitern versuchen. Es wurde im 4. Abschnitt gezeigt wie jede Darstellung $T(g)$ zu den von uns genannten „T — Distributionen" führt (die gewöhnlichen Distributionen sind dann L-Distributionen, wo L_g die reguläre Darstellung bedeutet). Dann erhält man Entwicklung allgemeiner Kerne nach T-Eigenkerne der selbstadjungierten Operatoren aus dem Zentrum des Lie'schen Ringes.

1. Allgemeine Eigenfunktionsentwicklungen

Wir beweisen zuerst das folgende wichtige

Lemma *). *Es sei $\Phi \subset H$ ein lokalkonvexer Raum und die Einschränkung der Fouriertransformation* (wegen Definitionen vgl. [9])

$$F: H \to \hat{H} = \int_{\Lambda} \hat{H}(\lambda)\, d\mu(\lambda)$$

sei nuklear, d. h:

(1) $$\Phi \ni \varphi \to \hat{\varphi} = F\varphi \in \hat{H}$$

ist nuklear. Dann ist die Abbildung

(2) $$\Phi \ni \varphi \to \hat{\varphi}(\lambda) \in (\hat{H}(\lambda)$$

für μ — f.a. $\lambda \in \Lambda$ stetig und sogar nuklear.

*) Den Beweis dieses Lemmas verdanke ich Lars Gårding.

Beweis. Da die Abbildung (1) nuklear ist, besitzt sie die folgende Form

$$\hat{\varphi} = \sum_k \langle \varphi, \varphi'_k \rangle \hat{h}_k \,, \quad \hat{h}_k \in \hat{H}; \tag{3}$$

φ'_k sind lineare stetige Funktionale auf dem Raume Φ, d. h. $\varphi'_k \in \Phi'$ und die Reihe

$$\sum_k \|\varphi'_k\|_{-p} \|\hat{h}_k\| < \infty \quad \text{für gewisses } p \,, \tag{4}$$

wo

$$\|\varphi'\|_{-p} \stackrel{\text{df}}{=} \sup_{|\varphi|_p \leqslant 1} |\langle \varphi, \varphi' \rangle| \,, \tag{5}$$

$\| \|_p$ bedeutet eine Seminorm in Φ. Jetzt normieren wir die $\hat{h}_k$: $\|\hat{h}_k\| = 1$, $k = 1, 2, \ldots$, dann haben wir also

$$\sum_k \|\varphi'_k\|_{-p} = \sum_k \|\varphi'_k\|_{-p} \|\hat{h}_k\|^2 < \infty \,. \tag{6}$$

Aus (6) und dem Satz von Fubini haben wir

$$\int_\Lambda \sum_k \|\varphi'_k\|_{-p} \|\hat{h}_k(\lambda)\|^2_\lambda \, d\mu(\lambda) < \infty \quad \text{d. h.}$$

$$\sum_k \|\varphi'_k\|_{-p} \|\hat{h}_k(\lambda)\|^2_\lambda < \infty \quad \text{für} \quad \mu - \text{f.a. } \lambda \in \Lambda \,. \tag{7}$$

Aus (5) und der Schwarz'schen Ungleichung folgt

$$\Big(\sum_k \langle \varphi, \varphi'_k \rangle \| \hat{h}_k(\lambda)\|_\lambda\Big)^2 \leqslant \Big(\sum_k \|\varphi\|_p \|\varphi'_k\|_{-p} \| \hat{h}_k(\lambda)\|_\lambda\Big)^2$$

$$= \|\varphi\|_p^2 \Big(\sum_k \big(\|\varphi'_k\|_{-p}^{1/2} \| \hat{h}_k(\lambda)\|_\lambda\big) \|\varphi'_k\|_{-p}^{1/2}\Big)^2 \leqslant \|\varphi\|_p^2 \sum_k \|\varphi'_k\|_{-p} \| \hat{h}_k(\lambda)\|^2_\lambda \sum_k \|\varphi'_k\|^2_{-p} = C\|\varphi\|_p^2 \,;$$

wegen (6) und (7) ist $C < \infty$.

Wir haben also gezeigt, dass für $\mu -$ f.a. $\lambda \in \Lambda$ sowohl

$$\hat{\varphi}(\lambda) \stackrel{\text{df}}{=} \sum \langle \varphi, \varphi_k \rangle \hat{h}_k(\lambda) \,, \tag{8}$$

wo $\hat{h}_k(\lambda) \in \hat{H}(\lambda)$, wobei

$$\sum_k \|\varphi'_k\|_{-p} \|\hat{h}_k(\lambda)_\lambda < \infty$$

als auch

$$\|\hat{\varphi}(\lambda)\|_\lambda \leqslant C\|\varphi\|_p \tag{9}$$

gilt.

Da aber (8) und (9) eben die Nuklearität und Stetigkeit der Abbildung (2) bedeuten, ist der Beweis vollständig erbracht.

As Korrollar haben wir augenblicklich (vgl. [9]) den

SATZ 1. *Es sei $\Phi \subset H$ ein lokalkonvexer Raum, wobei Φ eine dichte Untermenge von H bildet. Wenn die Einbettung (Injektion) $\Phi \ni \varphi \to \varphi \in H$ nuklear ist (wenn z. B. der Raum Φ selbst nuklear ist und die Einbettung $\Phi \ni \varphi \to \varphi \in H$ stetig ist), dann hat man für μ — f.a. $\lambda \in \Lambda$*

$$\hat{\varphi}_i(\lambda) = \langle \varphi, e_i(\lambda) \rangle ,$$

wo die Komponenten $e_i(\lambda)$ von $e(\lambda)$, $i = 1, 2, \ldots, \dim \hat{H}(\lambda)$ dem Raume Φ' angehören. Es gilt die Parseval'sche Gleichung

$$(10) \qquad (\varphi, \psi) = \int_\Lambda (\hat{\varphi}(\lambda), \hat{\psi}(\lambda))_\lambda d\mu(\lambda) = \int_\Lambda \sum_{i=1}^{\dim \hat{H}(\lambda)} \langle \varphi, e_i(\lambda) \rangle \overline{\langle \psi, e_i(\lambda) \rangle} d\mu(\lambda) .$$

Falls die Operatoren (A_ν) stetig auf Φ sind, dann sind $e_i(\lambda)$ ihre gemeinsame (verallgemeinerte) Eigenfunktionen:

$$\langle A_\nu \varphi, e_i(\lambda) \rangle = \hat{A}_\nu(\lambda) \langle \varphi, e_i(\lambda) \rangle ,$$

was man kürzer als

$$(11) \qquad A'_\nu e_i(\lambda) = \hat{A}_\nu(\lambda) e_i(\lambda) \quad \text{für} \quad \text{f. a. } \lambda \in \Lambda$$

schreiben kann.

Bemerkung. Auch wenn die Operatoren A_ν nicht stetig den Raum Φ in sich abbilden, bezeichnet man oft — im übertragenen Sinne — die Elemente $e_i(\lambda)$ als (verallgemeinerte) Eigenfunktionen von A_ν die zum Eigenwert $\hat{A}_\nu(\lambda)$ der Vielfachheit $\dim \hat{H}(\lambda)$ gehören.

2. Spektraldarstellung abstrakter Kerne

In diesem Abschnitt beweisen wir ein abstraktes Seitenstück zum Satz 1 der vorhergehenden Note [9].

SATZ 2. *Es sei 1° Φ ein nuklearer Raum; 2° die vertauschbaren Operatoren (A_ν) bilden den Raum Φ stetig in sich (vgl. jedoch die obige „Bemerkung"); 3° K sei eine positive hermitische bilineare Form auf Φ: $(\varphi, \psi) \overset{\text{df}}{=} K(\varphi, \psi)$; 4° Die Operatoren A_ν sind K-symmetrisch (allgemeiner K-normal)*

$$(A_\nu \varphi, \psi) \equiv (\varphi, A_\nu \psi) , \quad \varphi, \psi \in \Phi .$$

Es sei H die Vervollständigung von Φ im Sinne des skalaren Produktes $K(\varphi, \psi)$. Die Operatoren A_ν induzieren im Raume H vertauschbare Operatoren $\dot{A}_\nu$; sie sollen 5° selbstadjungierte (allgemeiner normale) vertauschbare Fortsetzungen $\bar{A}_\nu$ besitzen (im Folgenden werden wir die drei Arten von Operatoren $A_\nu, \dot{A}_\nu, \bar{A}_\nu$ mit A bezeichnen); es sei schliesslich 6° auf Φ eine

bilineare Form B definiert, welche im Raume Φ einen mit den A_ν vertauschbaren Operator B induziert:

$6^{\circ\prime}$ *B: $\Phi \to \Phi$ ist stetig.*

Statt $6^{\circ\prime}$ kann man die folgende nicht gleichwertige Voraussetzung machen

$6^{\circ\prime\prime}$ *B ist stetig in H.*

Behauptung: *Dann besitzt der Kern $B \in \Phi' \otimes \Phi'$ die folgende Spektraldarstellung*

$$B = \int_\Lambda B_\lambda d\mu(\lambda) \qquad \left(\text{genauer } B(\varphi, \psi) = \int_\Lambda B_\lambda(\varphi, \psi) d\mu(\lambda)\right).$$

Insbesondere besitzt der positivdefinite Kern K die Darstellung

$$K = \int_\Lambda K_\lambda d\mu(\lambda),$$

wobei die Kerne $B_\lambda (K_\lambda)$ Eigenkerne der Operatoren A_ν sind:

$$B_\lambda(\varphi, A_\nu \psi) = B_\lambda(A_\nu \varphi, \psi) = \hat{A}_\nu(\lambda) B_\lambda(\varphi, \psi) = {}^{\mathrm{tr}}B_\lambda(A_\lambda \psi, \varphi),$$

was man such kürzer als

$$A'_\nu B_\lambda = \hat{A}_\nu(\lambda) B_\lambda$$
$$A'_\nu {}^{\mathrm{tr}}B_\lambda = \hat{A}_\nu(\lambda) {}^{\mathrm{tr}}B_\lambda$$

schreiben kann.

Beweis. Aus dem Lemma und 6° folgt es, dass der Operator B für f.a. λ einen Kern $B_\lambda \in \Phi' \otimes \Phi'$ folgendermassen induziert:

$$\Phi \times \Phi \ni \{\varphi, \psi\} \to \left(\widehat{B\varphi}(\lambda), \hat{\psi}(\lambda)\right)_\lambda = B_\lambda(\varphi, \psi).$$

Die Behauptung folgt jetzt unmittelbar aus dem v. Neumann'schen Spektralsatz (vgl. die vorhergehende Note [9]):

$$B(\varphi, \psi) = (B\varphi, \psi) = \int \left(\widehat{B\varphi}(\lambda), \hat{\psi}(\lambda)\right)_\lambda d\mu(\lambda)$$
$$= \int B_\lambda(\varphi, \psi) d\mu(\lambda) = \int \langle \varphi, B_\lambda \psi \rangle d\mu(\lambda).$$

Da für f.a. λ gilt

$$B_\lambda(A_\nu \varphi, \psi) = \left((BA_\nu \varphi)\hat{}(\lambda), \hat{\psi}(\lambda)\right)_\lambda = \left((A_\nu B\varphi)\hat{}(\lambda), \hat{\psi}(\lambda)\right)_\lambda$$
$$= \hat{A}_\nu(\lambda)\left((B\varphi)\hat{}(\lambda), \hat{\psi}(\lambda)\right)_\lambda = \hat{A}_\nu(\lambda) B_\lambda(\varphi, \psi)$$
$$= \left((B\varphi)\hat{}(\lambda), (A_\nu \psi)\hat{}(\lambda)\right)_\lambda = B_\lambda(\varphi, A_\nu \psi)$$
$$= {}^{\mathrm{tr}}B_\lambda(A_\nu \psi, \varphi).$$

3. Konstruktion des nuklearen Raumes Φ

In diesem Abschnitt zeigen wir, wie die obig aufgezeigte Situation im Falle abzählbar vielen vertauschbaren Operatoren (A_j) herbeigeführt werden kann. Die hier gegebene Konstruktion sichert also erstens die Existenz eines vollständigen Systems der (verallgemeinerten) Eigenfunktionen, zweitens gibt die Spektraldarstellung der Kerne im Falle eines gegebenen vertauschbaren Systems normaler Operatoren in einem abstrakten (separablen) Hilbert'schen Raume *). Es gilt nämlich der

Satz 3. *Es sei* $(A_j)_1^\infty$ *ein System der Operatoren, deren Definitionsbereiche* $D(A_j)$ *einen dichten Durchschnitt* $D_0 = \bigcap_{j,p} D(A_j^p)$ *besitzen. Dann existiert ein nuklearer Raum* Φ*, der die in den Sätzen* 1 *und* 2 (1°-4°) *aufgezählten Eigenschaften besitzt.*

Beweis (Konstruktion). Es sei $0 \neq v_j \in D_0$. Wir bilden den endlichdimensionalen Unterraum

$$\Phi_{j;m,N}^0 \overset{\text{df}}{=} \left\{ u = \sum_{|\alpha| \leqslant m} a_\alpha A^\alpha v_j \right\},$$

wo wir zur Abkürzung $A^\alpha \overset{\text{df}}{=} A_1^{\alpha_1} \dots A_N^{\alpha_N}$, $|\alpha| = \alpha_1 + \dots + \alpha_N$ gesetzt haben. Den Raum topologiesieren wir folgendermassen: $\overset{n}{u} = \sum_{|\alpha| \leqslant m} \overset{n}{a}_\alpha A^\alpha v_j \to 0$ (dann und nur dann) wenn

$$\max_\alpha |\overset{n}{a}_\alpha| \to 0 , \qquad n \to \infty .$$

Wenn nötig, dividieren wir durch den Nullunterraum und vervollständigen zu einem nuklearen (da endlichmensionalen) Raum $\Phi_{j;m,N}$. Jetzt bilden wir den induktiven Limes Φ_j der Räume $\Phi_{j;m,N}$ $m, N \to \infty$. Φ_j ist nuklear, als induktiver Limes der nuklearen Räume (vgl. [10]). Wenn Φ_1 dicht in D_0 ist, sind wir am Ziel (vgl. den nachstehenden "Beispiel"). Wenn nicht, nehmen wir einen Vektor $\Phi_1 \not\ni v_2 \in D_0$ und durch die Wiederholung der vorhergehenden Konstruktion bekommen wir den Raum Φ_2. Wir bilden die direkte Summe $\Phi_1 + \Phi_2$ (mit der üblichen Topologie). Wenn nötig, wiederholen wir diesen Prozess und bekommen zum Schluss den verlangten Raum

$$\Phi = \sum \Phi_j .$$

Als abzählbare direkte Summe der nuklearen Räume ist Φ nuklear (vgl. [10]). Selbstredend besitzt er alle verlangte Eigenschaften.

Beispiel. Es sei G eine diskrete abel'sche (lokalkompakte) Gruppe und es seien $A_j \overset{\text{df}}{=} L_{g_j}$ die Verschiebungsoperatoren. Als v_1 nehmen wir

*) Diese Konstruktion verdanke ich Jan Odhnoff.

die Funktion $v_1 = \delta_e(\cdot)$, d.h. das im neutralen Element e der Gruppe G konzentriertes Einheitsmass. Dann, wie man leicht sehen kann, ist $\Phi = \Phi_1$ der Raum der (stetigen, beschränkten) Funktionen mit kompakten Trägern mit der üblichen Topologie: die Folge $\varphi_n \to 0$ wenn die Träger einer kompakten (hier endlichen) Menge angehören und die Folge konvergiert gleichmässig gegen Null. In diesem Falle sind auch $\varphi' \in \Phi'$ Funktionen und durch Anwendung der Sätze 1 und 2 bekommen wir einen neuen Beweis von Satz 4 in [9].

4. Eine Verallgemeinerung der Distributionen auf Lie'schen Gruppen

In diesem Abschnitt zeigt man, wie eine stetige Darstellung

$$G \ni g \to T(g) \in \mathcal{L}(H, H)$$

der Lie'schen Gruppe G in einen separablen Hilbert-(Banach oder einen noch allgemeineren lokalkonvexen Raum) Raum H einen nuklearen Raum Φ induziert. Dass der duale Raum Φ' als der Raum der verallgemeinerten Distributionen d.s.g. "T-Distributionen" betrachtet werden darf zeigt das nachstehende Beispiel.

Es sei $\mathfrak{G}$ der (in H dichte) Gårding-Unterraum der Darstellung $g \to T(g)$, d.h.:

$$\mathfrak{G} \overset{\text{df}}{=} \left\{ u \in H\colon\ u = T_\psi v \overset{\text{df}}{=} \int_G T(g)\psi(g)v\,dg\ ,\ v \in H\ ,\ \psi \in \mathcal{D}(G) \right\},$$

dg bedeutet das Haarsche Mass auf G.

Nur der erste Schritt der Konstruktion des nuklearen Raumes $\Phi \subset H$ weicht von der des vorhergehenden Abschnittes:

$$\Phi_j^0 \overset{\text{df}}{=} \{T_\varphi v_j;\ v_j \in H\}\ .$$

Φ_j^0 wird topologiesiert durch die folgende Übertragung der Schwartz'schen Topologie von $\mathcal{D}(G)$:

$$\overset{n}{u} = T_{\varphi_n} v_j \underset{n\to\infty}{\to} 0 \quad \text{dann und nur dann wenn } \varphi_n \to 0 \text{ in } \mathcal{D}(G)\ .$$

Durch die eventuelle Division durch den Nullunterraum und die Vervollständigung bekommen wir den nuklearen Raum Φ_j (weil $\mathcal{D}(G)$ nuklear ist). Der verlangte Raum Φ ist wieder eine (höchstens abzählbare) direkte Summe der nuklearen Räume Φ_j: $\Phi \overset{\text{df}}{=} \sum_j \Phi_j$. Natürlich ist der Raum Φ dicht in $\mathfrak{G}$ also auch in H. Die Topologie in Φ ist stärker als die Topologie von H. Wir führen jetzt die folgende

DEFINITION. *Es sei $T(g)$ eine stetige Darstellung der Lie'schen Gruppe G. Die Elemente von Φ',* d.h *die stetigen linearen Funktionale auf dem eben konstruierten nuklearen Raume Φ werden T — Distributionen — die Elemente von $\Phi' \otimes \Phi'$ werden T — Kerne genannt.*

Beispiel. Wenn man als H den Raum $L^2(G)$, $v_j \in C_0(G)$ (stetige Funktionen mit kompakten Trägern) und als $T(g)$ die reguläre Darstellung L_g nimmt, dann ist $T_\psi v_j = L_\psi v_j = \psi * v_j$ eine beliebig oft differenzierbare Funktion mit kompaktem Träger. Die Konvergenz in Φ_j (also in Φ) ist die Schwartz'sche Konvergenz. Die L – Distributionen sind also in diesem Falle gewöhnliche Distributionen auf der Lie'schen Gruppe G.

Wie Gårding in [6] gezeigt hat, gilt für die infinitesimalen Erzeugenden X_k der 1-gliedrigen Untergruppe $T(\exp te_k)$ die folgende Formel

$$X_k T_\psi v = T_{\partial\psi/\partial e_k} v\,,$$

wo $\partial/\partial e_k$ die Differentiation in Richtung der 1-gliedrigen Gruppe $\exp(te_k)$ bedeutet. Daraus folgt, dass die Operatoren X_k in jedem Raume Φ_j – also im ganzen nuklearen Raume Φ – stetig sind. Deswegen bilden die symmetrischen Polyonome A_j im Zentrum des Lie'schen Ringes (vgl. [8]) den Raum Φ stetig in sich:

$$A_j\colon \Phi \to \Phi \quad \text{stetig}\,.$$

Bekanntlich sind die A_j wesentlich selbstadjungiert (vgl [8]) und induzieren eine Fouriertransformation

$$H \to \hat{H} = \int_\Lambda \hat{H}(\lambda)\, d\mu(\lambda)\,.$$

Wieder sind die Voraussetzungen der Sätze 1 und 2 erfüllt. Wir bekommen also als Korrollare abstrakte Seitenstücke von den Sätzen der früheren Abhandlungen des Verfassers [8], [9]:

Satz 4. *Es sei $G \ni g \to T(g) \in \mathcal{L}(H, H)$ eine (abstrakte) unitäre Darstellung der separablen zusammenhängenden Lie'schen Gruppe G im Raume H. Dann besitzt das System (A_j) der symmetrischen Operatoren aus dem Zentrum des Lie'schen Ringes einen vollständigen System der verallgemeinerten Eigenelemente $(e_k(\lambda))$ $k = 1, \ldots, \dim \hat{H}(\lambda)$. Es gilt sowohl die Parseval'sche Formel* (10) *als auch* (11). *Die $e_k(\lambda)$ sind für* f.a. $\lambda \in \Lambda$ *T-Distributionen.*

Satz 5. *Einen den Voraussetzungen des Satzes* 2 *genügenden Kern $B \in \Phi' \otimes \Phi'$ kann man nach den T-Eigenkernen B_λ von A_j entwickeln:*

$$\Phi' \otimes \Phi' \ni B = \int_\Lambda B_\lambda d\mu(\lambda)\,, \quad \text{wo} \quad B_\lambda \in \Phi' \otimes \Phi'$$

$$A_j' B_\lambda = \hat{A}_j(\lambda) B_\lambda \quad \text{für} \quad \mu - \text{f.a. } \lambda\,.$$

Diese Abhandlung enstand in Lund (Schweden) wo ich 3 Monate als Rockefeller-Stipendist verbringen durfte. Von unschätzbarer Bedeutung waren für mich Diskussionen mit Prof. Dr Lars Gårding und Dr Jan Odhnoff, denen ich hier meinen herzlichsten Dank aussprechen möchte.

Bemerkung bei Korrektur. Soeben erschien eine interessante Abhandlung von Ju. M. Berezanskij: "Über Eigenfunktionsentwicklungen der selbstadjungierten Operatoren" Ukr. Mat. Zhurn., **11** (1959), 16-24 (russisch), die sich in dem Gedankenkreise der Abhandlung [7] von G. I. Kac bewegt.

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SCHRIFTTUM

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MATHEMATICS

On the Infinite Product Representation of Functions Regular and Nonvanishing in the Unit Circle

by

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Presented by K. BORSUK on June 2, 1959

Dedicated to G. Alexits on his 60th birthday

1. Let $f(z)$ be regular in a neighbourhood A of the origin and $f(0) = 1$. As pointed out by J. Ritt [1], $f(z)$ can be expanded in the product

$$\prod_{\nu=1}^{\infty}(1+a_\nu z^\nu) \tag{1.1}$$

representing $f(z)$ in a subregion B of A, containing the origin. The determination of his a_ν's is rather clumsy and the determination of the exact domain of convergence has been left open. Another product-representation, as Vera T. Sós remarked, can, for $f(z)$ regular and nonvanishing for $|z| \leq 1$, be given by expanding $\log f(z)$ (with $\log f(0) = 0$) in the power-series

$$\log f(z) = \sum_{\nu=1}^{\infty} b_\nu z^\nu, \tag{1.2}$$

and then

$$f(z) = \prod_{\nu=1}^{\infty} (e^{z^\nu})^{b_\nu} \tag{1.3}$$

is another type of product-representation. The replacement of $e^{b_\nu z^\nu}$ by the first two terms of its Taylor-series was perhaps the starting point of Ritt's investigation.

If e. g. $\log f(z) \in C$ *), then the series (1.2) is (both with its real and imaginary parts) generally divergent on some set on $|z| = 1$ (the same holds for (1.3)). The usual way to avoid such difficulties is to pass on

*) The class C consists of functions regular for $|z| < 1$ vanishing for $z = 0$ and continuous in $|z| \leq 1$.

to summability methods instead of convergence. However, such difficulties can at least partly be avoided by using another representation for $\log f(z)$ (and generally to functions analytic in $|z|<1$), this expansion having also other advantages. For this reason we form the following sequence of rational functions of z. Let

$$f_{00}(z)\equiv 1\,,\qquad f_{11}(z)=1+\frac{2z}{1-z}$$

and for $\nu\geq 2$, $k=1,2,\ldots,2^{\nu-1}$ with

$$F_\nu(z)=1-\frac{4z\sin^2\frac{\pi}{2^\nu}}{1-2z\cos\frac{2\pi}{2^\nu}+z^2}\,. \tag{1.4}$$

let us write the functions

$$f_{\nu k}(z)=F_\nu(ze^{2\pi i\frac{2k-1}{2^\nu}})\,. \tag{1.5}$$

Since, as is easily seen, $F_\nu(z)$ can be written also in the form

$$F_\nu(z)=\frac{(1-z)^2}{(1-ze^{\frac{2\pi i}{2^\nu}})(1-ze^{\frac{-2\pi i}{2^\nu}})}\,, \tag{1.6}$$

the functions $\log f_{\nu k}(z)$ with $\log f_{\nu k}(0)=0$ are well defined for $|z|<1$ (and by radial limes for $|z|\leq 1$). We shall also write this sequence $f_{\nu k}(z)$ (in the above order) without second indices as

$$f_0(z),\ f_1(z),\ \ldots,\ f_n(z),\ \ldots$$

and put forward the following

THEOREM. *If $F(z)\in C$, then the real constants d_n can be determined so that the series* *)

$$\sum_{n=1}^{\infty} d_n\log f_n(z) \tag{1.7}$$

converges to $F(z)$ in $|z|<1$, uniformly in $|z|\leq 1-\varepsilon$, $\varepsilon>0$. The series

$$\sum_{n=1}^{\infty} d_n\operatorname{Im}\log f_n(z) \tag{1.8}$$

converges for $|z|\leq 1$ uniformly to $\operatorname{Im}F(z)$.

In particular, if $F(z)=\log f(z)$ and $f(z)$ is regular and non-vanishing in $|z|\leq 1$ with **) $f(0)=1$, then we obtain with real d_n's the product-

*) Let us note that, from (1.6) it easily follows that the functions $\log f_n(z)$ are univalent for $|z|<1$.

**) These conditions could have been essentially relaxed.

representation

$$f(z) = \prod_{n=1}^{\infty} f_n(z)^{d_n} \tag{1.9}$$

with the rational functions $f_n(z)$ in (1.4)-(1.5) converging uniformly for $|z| \leq 1-\varepsilon$.

The proof of the convergence of the series

$$\sum_{n=0}^{\infty} d_n \operatorname{Re} \log f_n(z) = \sum_{n=0}^{\infty} d_n \log |f_n(z)| \tag{1.10}$$

on the whole unit-circle is illusory since the functions $\log|f_n(z)|$ become infinite on a dense set on $|z| = 1$. This shows a very unusual behaviour of the expansion (1.7). The question of convergence of (1.10) for other z's seems to be a very delicate one; to this, to the construction of a "better" conjugate orthogonal *) system and to a general treatment of such systems I shall return elsewhere as well as to the treatment of the analogous questions concerning more general domains.

2. An essential role is played in the investigation by real orthonormal systems the general theory of which owes very much to the papers of G. Alexits and his pupils. Among the many orthonormal systems, which could have been used for our purposes, perhaps the one leading to the simplest formulas is the Haar-system [2], which must be slightly modified, being defined on the unit-circle instead of the interval $[0, 1]$. Let

$$\chi_0(e^{2\pi i x}) \equiv 1 \quad \text{for} \quad 0 \leq x \leq 1 ,$$

further

$$\begin{aligned} &\chi_{11}(e^{2\pi i x}) = 1 \quad \text{for} \quad 0 < x < \tfrac{1}{2} \\ &\chi_{11}(e^{2\pi i x}) = -1 \quad \text{for} \quad \tfrac{1}{2} < x < 1 \end{aligned} \tag{2.1}$$

and **) for $x = 0, \frac{1}{2}$

$$\chi_{11}(e^{2\pi i x}) = \tfrac{1}{2}\{\chi_{11}(e^{2\pi i(x+0)}) + \chi_{11}(e^{2\pi i(x-0)})\} .$$

For $\nu \geq 2$ and $1 \leq k \leq 2^{\nu-1}$ we define

$$\begin{aligned} &\chi_{\nu k}(e^{2\pi i x}) = 0 \quad \text{for} \quad 0 < x < \frac{2k-2}{2^{\nu}} , \\ &\chi_{\nu k}(e^{2\pi i x}) = 2^{\frac{\nu-1}{2}} \quad \text{for} \quad \frac{2k-2}{2^{\nu}} < x < \frac{2k-1}{2^{\nu}} , \\ &\chi_{\nu k}(e^{2\pi i x}) = -2^{\frac{\nu-1}{2}} \quad \text{for} \quad \frac{2k-1}{2^{\nu}} < x < \frac{2k}{2^{\nu}} , \end{aligned} \tag{2.2}$$

*) It is easy to show that the sequence $\log|f_n(e^{2\pi i x})|$ forms in $[0,1]$ also an orthonormal system.

**) In his original form Haar put $\chi_{11}(0) = 1$ and $\chi_{11}(1) = -1$.

and for $x = 0, \frac{2k-2}{2^\nu}, \frac{2k-1}{2^\nu}, \frac{2k}{2}$

$$\chi_{\nu k}(e^{2\pi i x}) = \tfrac{1}{2}\{\chi_{\nu k}(e^{2\pi i(x+0)}) + \chi_{\nu k}(e^{2\pi i(x-0)})\}.$$

We shall sometimes denote the sequence $\chi_{\nu k}(e^{2\pi i x})$ (in the above order without second indices) by

$$\chi_0(e^{2\pi i x}), \chi_1(e^{2\pi i x}), \ldots, \chi_n(e^{2\pi i x}), \ldots \tag{2.3}$$

The orthonormality of this system is meant in the sense

$$\int_0^1 \chi_\mu(e^{2\pi i x})\chi_\nu(e^{2\pi i x})\,dx = \begin{matrix} 0 & \text{for} & \mu \neq \nu \\ 1 & & \mu = \nu \end{matrix}. \tag{2.4}$$

The basic observation is that, for $0 \leq x \leq 1$,

$$\nu = 2, 3, \ldots \qquad k = 1, 2, \ldots, 2^{\nu-1},$$

the relation

$$\lim_{r\to 1-0} \frac{2^{\frac{\nu-1}{2}}}{\pi} \cdot \operatorname{Im}\log f_{\nu k}(re^{2\pi i x}) = \chi_{\nu k}(e^{2\pi i x}) \tag{2.5}$$

holds as well as

$$\lim_{r\to 1-0} \frac{2}{\pi} \operatorname{Im}\log f_{11}(re^{2\pi i x}) = \chi_{11}(e^{2\pi i x}). \tag{2.6}$$

3. Writing $F(e^{2\pi i x}) = U(x) + iV(x)$ and

$$c_n = \int_0^1 V(x)\chi_n(e^{2\pi i x})\,dx \tag{3.1}$$

we assert that with these (real) c_n's the d_n-coefficients of the Theorem can be given as

$$d_1 = \frac{2}{\pi} c_1 \tag{3.2}$$

and, for $n \geq 2$, with the obvious notation

$$d_n = d_{\nu k} = \frac{2^{\frac{\nu-1}{2}}}{\pi} c_{\nu k}. \tag{3.3}$$

We have to consider with these d_n's the series

$$\sum_{n=1}^{\infty} d_n \log f_n(z)$$

for $|z| < 1$. We shall use the known fact that if $G(z)$ is regular in $|z| \leq r$ and $G(0) = 0$ then, for $|z| < r$, we have

$$G(z) = i \int_0^1 \frac{r + ze^{-2\pi i\vartheta}}{r - ze^{-2\pi i\vartheta}} \operatorname{Im} G(re^{2\pi i\vartheta})\,d\vartheta. \tag{3.4}$$

When applied first with $G(z) = F(z)$ and with $r = 1$ this gives

$$F(z) = i \int_0^1 \frac{1+ze^{-2\pi i\vartheta}}{1-ze^{-2\pi i\vartheta}} V(\vartheta)\, d\vartheta \,. \tag{3.5}$$

Applying (3.4) with $G(z) = \log f_{\nu k}(z)$ and $|z| < r (< 1)$ we get

$$\log f_{\nu k}(z) = i \int_0^1 \frac{r+ze^{-2\pi i\vartheta}}{r-ze^{-2\pi i\vartheta}} \operatorname{Im} \log f_{\nu k}(re^{2\pi i\vartheta})\, d\vartheta \,,$$

since the integrand converges boundedly to its limit and owing to (2.5), for $\nu \geq 2$,

$$\frac{2^{\frac{\nu-1}{2}}}{\pi} \log f_{\nu k}(z) = i \int_0^1 \frac{1+ze^{-2\pi i\vartheta}}{1-ze^{-2\pi i\vartheta}} \chi_{\nu k}(e^{2\pi i\vartheta})\, d\vartheta$$

or from (3.3)

$$d_{\nu k} \log f_{\nu k}(z) = i \int_0^1 \frac{1+ze^{-2\pi i\vartheta}}{1-ze^{-2\pi i\vartheta}} c_{\nu a}\, \chi_{\nu k}(e^{2\pi i\vartheta})\, d\vartheta \,.$$

For $\nu = 1$ we have from (2.7)

$$\frac{2}{\pi} \log f_{11}(z) = i \int_0^1 \frac{1+ze^{-2\pi i\vartheta}}{1-ze^{-2\pi i\vartheta}} \chi_{11}(e^{2\pi i\vartheta})\, d\vartheta \,,$$

i. e.

$$d_{11} \log f_{11}(z) = i \int_0^1 \frac{1+ze^{-2\pi i\vartheta}}{1-ze^{-2\pi i\vartheta}} c_{11}\, \chi_{11}(e^{2\pi i\vartheta})\, d\vartheta$$

and hence, from all these for all integers $N \geq 1$

$$F(z) - \sum_{n=1}^{N} d_n \log f_n(z) = i \int_0^1 \frac{1+ze^{-2\pi i\vartheta}}{1-ze^{-2\pi i\vartheta}} \left\{ V(\vartheta) - \sum_{n=1}^{N} c_n \chi_n(e^{2\pi i\vartheta}) \right\} d\vartheta \,. \tag{3.6}$$

Taking into account that

$$-i \int_0^1 \frac{1+ze^{-2\pi i\vartheta}}{1-ze^{-2\pi i\vartheta}} c_0 \chi_0(e^{2\pi i\vartheta})\, d\vartheta = -c_0 i = -i \int_0^1 V(\vartheta)\, d\vartheta$$

$$= -i \int_0^1 \operatorname{Im} F(e^{2\pi i\vartheta})\, d\vartheta = -i \operatorname{Im} F(0) = 0$$

we get from (3.6) and (3.2)

$$(3.7)\qquad R_N(z) \overset{\text{def}}{=} F(z) - \sum_{n=1}^{N} d_n \log f_n(z)$$

$$= i \int_0^1 \frac{1+ze^{-2\pi i\vartheta}}{1-ze^{-2\pi i\vartheta}} \left\{ V(\vartheta) - \sum_{n=0}^{N} c_n \chi_n(e^{2\pi i\vartheta}) \right\} d\vartheta .$$

But, for $|z| \leq 1-\varepsilon$, Cauchy's inequality gives

$$|R_N(z)| \leq \frac{1}{\varepsilon} \sqrt{\int_0^1 \left(V(\vartheta) - \sum_{n=0}^{N} c_n \chi_n(e^{2\pi i\vartheta})\right)^2 d\vartheta} ,$$

which tends to 0 with $1/N$ owing to the completeness of the Haar-system.

In orde to prove the second assertion of the Theorem we have to remark that owing te [2] since $V(\vartheta)$ is continuous, the series converges uniformly for $0 \leq \vartheta \leq 1$ to $V(\vartheta)$, i. e. for $\varepsilon > 0$ and $m > n > n_0(\varepsilon)$, $0 \leqslant \vartheta \leqslant 1$

$$\left| \sum_{\nu=n}^{m} d_n \operatorname{Im} \log f_\nu(e^{2\pi i\vartheta}) \right| \leq \varepsilon .$$

But owing to the maximum-principle, we have *a fortiori* for $|z| \leq 1$

$$\left| \sum_{\nu=n}^{m} d_\nu \operatorname{Im} \log f_\nu(z) \right| \leq \varepsilon ,$$

which proves the assertion.

I am indebted for valuable remarks to Vera T. Sós and J. Czipszer, to the latter also for a simplification of my original proof.

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MATHEMATICS

Estimates of the Exponential Growth of Solutions of a Second Order Ordinary Differential Equation

by

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1. We shall consider the differential equation

$$x''+2a(t)x'+b(t)x=0\,. \tag{1}$$

Assume that $a(t)$ and $b(t)$ are piecewise continuous and that for $t_0 \leqslant t < +\infty$ the following inequalities hold

$$a_1 \leqslant a(t) \leqslant a_2\,, \qquad b_1 \leqslant b(t) \leqslant b_2\,, \qquad a_i, b_i \text{ are const. } (i=1,2)\,. \tag{2}$$

PROBLEM P. This problem consists in finding two constants $\varrho_1 \leqslant \varrho_2$ depending on a_i, b_i such that: 1° for arbitrary functions $a(t)$ and $b(t)$ satisfying (2) and every nontrivial solution $x(t)$ of (1), the inequalities,

$$\limsup_{t\to+\infty}|x(t)|\exp\bigl(-(\varrho_1-\varepsilon)t\bigr)>0 \qquad \text{and} \qquad \limsup_{t\to+\infty}|x(t)|\exp\bigl(-(\varrho_2+\varepsilon)t\bigr)<+\infty \tag{3}$$

hold for any $\varepsilon>0$, and 2° for some functions $a(t)$ and $b(t)$ satisfying (2), there exists a solution $x(t)$ of (1) such that the first (second) inequality of (3) is not satisfied whenever $\varepsilon<0$.

Denote by Δ the minimum of a^2-b for $a_1 \leqslant a \leqslant a_2$, $b_1 \leqslant b \leqslant b_2$. In the treatment of Problem P we distinguish two cases:

(i) $\Delta<0$ and (ii) $\Delta \geqslant 0$.

Using the same method as in [1] (in paper [1] Problem P has been considered in the case $a(t)=0$, $b_1>0$) we give in the present paper the solution of Problem P in the case (i). These results have been presented at the Seminar of Professor Ważewski on October 29th, 1958. The solution of Problem P in the case (ii) results immediately from the author's paper [2] *).

*) An analogous problem was treated by T. Tatarkiewicz in a lecture given at the Polish Mathematical Society, Cracow Branch, on February 17th, 1959.

2. Instead of (1) we consider the equivalent system $x' = y$, $y' = -b(t)x - 2a(t)y$. We apply the transformation

$$x = u\exp(\varrho t), \quad y = v\exp(\varrho t) \tag{4}$$

and we obtain the system

$$u' = -\varrho u + v, \quad v' = -b(t)u - (2a(t) + \varrho)v. \tag{5}$$

Remark 1. If for $\varrho = \varrho_0$, system (5) does not admit any non-trivial solution converging to zero as $t \to +\infty$, then $\varrho_0 \leqslant \varrho_1$. Similarly, if every solution of (5) for $\varrho = \varrho_0$ is bounded for $t_0 \leqslant t < +\infty$, then $\varrho_0 \geqslant \varrho_2$. Hence, Problem P will be solved if we determine ϱ_1 and ϱ_2 in such a manner that: 1° for arbitrary $a(t)$ and $b(t)$ satisfying (2), each nontrivial solution of (5) does not approach zero as $t \to +\infty$ whenever $\varrho < \varrho_1$ and is bounded if $\varrho > \varrho_2$, and 2° for some functions $a(t)$ and $b(t)$ satisfying (2), system (5) admits for $\varrho > \varrho_1$ solutions converging to zero or for $\varrho < \varrho_2$ unbounded solutions.

3. We now introduce two auxiliary systems (6) and (7):

$$u' = -\varrho u + v, \quad v' = \sigma_1(u, v), \tag{6}$$

where

$$\begin{aligned}
&\sigma_1(u, v) = -b_2 u - (2a_2 + \varrho)v, && \text{if} \quad uv \geqslant 0, \quad (v - \varrho u)v \geqslant 0, \quad v - \varrho u \neq 0,\\
&\sigma_1(u, v) = -b_1 u - (2a_1 + \varrho)v, && \text{if} \quad \varrho > 0 \quad \text{and} \quad (v - \varrho u)v < 0,\\
&\sigma_1(u, v) = -b_2 u - (2a_1 + \varrho)v, && \text{if} \quad \varrho < 0 \quad \text{and} \quad (v - \varrho u)v \leqslant 0,\\
&\sigma_1(u, v) = -b_1 u - (2a_2 + \varrho)v, && \text{if} \quad uv \leqslant 0, \quad (v - u)v \geqslant 0, \quad v - \varrho u \neq 0.
\end{aligned}$$

$$u' = -\varrho u + v, \quad v' = \sigma_2(u, v), \tag{7}$$

where

$$\begin{aligned}
&\sigma_2(u, v) = -b_1 u - (2a_1 + \varrho)v, && \text{if} \quad uv \geqslant 0, \quad (v - \varrho u)v \geqslant 0, \quad v - u \neq 0,\\
&\sigma_2(u, v) = -b_2 u - (2a_2 + \varrho)v, && \text{if} \quad \varrho > 0 \quad \text{and} \quad (v - \varrho u)v < 0,\\
&\sigma_2(u, v) = -b_1 u - (2a_2 + \varrho)v, && \text{if} \quad \varrho < 0 \quad \text{and} \quad (v - \varrho u)v \leqslant 0,\\
&\sigma_2(u, v) = -b_2 u - (2a_1 + \varrho)v, && \text{if} \quad uv \leqslant 0, \quad (v - \varrho u)v \geqslant 0, \quad v - \varrho u \neq 0.
\end{aligned}$$

The right-hand members of these systems do not depend on t, and, in general, they are not continuous on the line $v = \varrho u$. Neverthless we will consider the solutions of (6) and of (7) passing through the line $v = \varrho u$, that is the pairs of continuous functions $u(t), v(t)$ such that for $0 < |t - t_*| < \varepsilon$ they satisfy system (6) (or (7)) and the inequality $(v(t) - \varrho u(t))(t - t_*) < 0$, and for $t = t_*$ we have $v(t_*) - \varrho u(t_*) = 0$. It may occur that system (6) (or (7)) does not admit such solutions. The following conditions guarantee the existence of solutions of (6) (or of (7)) passing through the line $v = \varrho u$:

$$u_0 \lim_{v \to \varrho u_0 + 0} \sigma_i(u_0, v) < 0 \quad \text{and} \quad u_0 \lim_{v \to \varrho u_0 - 0} \sigma_i(u_0, v) < 0, \tag{8}$$

where $u_0 \neq 0$, $i = 1, 2$.

Due to the definition of $\sigma_i(u, v)$ the conditions (8) are equivalent for $i = 1$, as well as for $i = 2$, to the inequalities

$$(9) \qquad \varrho^2 + 2a_2\varrho + b_2 > 0\,, \quad \varrho^2 + 2a_1\varrho + b_1 > 0\,, \quad \text{if} \quad \varrho > 0\,,$$

$$(10) \qquad \varrho^2 + 2a_1\varrho + b_2 > 0\,, \quad \varrho^2 + 2a_2\varrho + b_1 > 0\,, \quad \text{if} \quad \varrho \leqslant 0\,.$$

Further, since we have

$$(11) \qquad \sigma_i(cu, cv) = c\sigma(u, v)\,, \quad \text{where } c \text{ is const., } i = 1, 2\,,$$

we conclude that, if $u(t), v(t)$ is a solution of (6) (or of (7)), then $cu(t)$, $cv(t)$ is also a solution of (6) (or of (7)). In particular, $-u(t), -v(t)$ satisfies system (6) (or (7)). Therefore, if one of the solutions of (6) (or of (7)) is oscillatory or periodic, then all other solutions have the same property.

The following inequalities follow from (2)

$$(12) \qquad \sigma_1(u, v) \leqslant -b(t)u - (2a(t) + \varrho)v \leqslant \sigma_2(u, v)\,, \quad \text{if} \quad v - \varrho u > 0\,,$$

$$(13) \qquad \sigma_2(u, v) \leqslant -b(t)u - (2a(t) + \varrho)v \leqslant \sigma_1(u, v)\,, \quad \text{if} \quad v - \varrho u < 0\,.$$

4. In this section we give sufficient and necessary conditions for the solutions of (6) (or of (7)) to be oscillatory. Consider the system

$$(14) \qquad u' = -\varrho u + v\,, \quad v' = -bu - (2a + \varrho)vf, \quad (a, b - \text{const.})\,.$$

It is easy to verify that the solution of (14) originating on the half-line $v = \beta u$, $u > 0$ reaches in a finite time the half-line $v = \alpha u$, $u > 0$ ($\alpha < \beta$), if and only if $\lambda^2 + 2a\lambda + b > 0$, for $\alpha \leqslant \lambda \leqslant \beta$. On the basis of this remark we conclude that the solutions of (6) are oscillatory, if and only if the following inequalities hold

$$(15_1) \quad \lambda^2 + 2a_2\lambda + b_2 > 0 \quad \text{for} \quad \lambda \geqslant \varrho\,, \quad \text{if} \quad \varrho > 0 \quad \text{and for} \quad \lambda \geqslant 0\,, \quad \text{if} \quad \varrho < 0\,,$$

$$(15_2) \quad \lambda^2 + 2a_1\lambda + b_1 > 0 \quad \text{for} \quad 0 \leqslant \lambda \leqslant \varrho\,, \quad \text{if} \quad \varrho > 0\,,$$

$$(15_3) \quad \lambda^2 + 2a_1\lambda + b_2 > 0 \quad \text{for} \quad \varrho \leqslant \lambda \leqslant 0\,, \quad \text{if} \quad \varrho \leqslant 0\,,$$

$$(15_4) \quad \lambda^2 + 2a_2\lambda + b_1 > 0 \quad \text{for} \quad \lambda \leqslant 0\,, \quad \text{if} \quad \varrho \geqslant 0 \quad \text{and for} \quad \lambda \leqslant \varrho\,, \quad \text{if} \quad \varrho < 0\,.$$

Denote by $\lambda_i(a_j, b_k)$ $(i = 1, 2)$ the roots of the equation $\lambda^2 + 2a_j\lambda + b_k = 0$ $(j, k = 1, 2)$. If $\lambda_i(a_j, b_k)$ are real-valued, then we assume that $\lambda_1(a_j, b_k) \leqslant \lambda_2(a_j, b_k)$.

It may be easily seen that the inequalities (15_1)-(15_4) are equivalent to the following conditions, respectively:

$$(16_1) \quad \lambda_2(a_2, b_2) \text{ is an imaginary number or } \max(0, \varrho) > \lambda_2(a_2, b_2)\,,$$

$$(16_2) \quad \lambda_2(a_1, b_1) \text{ is an imaginary number or } \varrho < \lambda_1(a_1, b_1) \text{ or } 0 > \lambda_2(a_1, b_1)$$

$$(16_3) \quad \lambda_1(a_1, b_2) \text{ is an imaginary number or } \varrho > \lambda_2(a_1, b_2) \text{ or } 0 < \lambda_1(a_1, b_2)$$

$$(16_4) \quad \lambda_1(a_2, b_1) \text{ is an imaginary number or } \min(0, \varrho) < \lambda_1(a_2, b_1)\,.$$

The conditions (16_1) and (16_3) follow from (i). Indeed, by (i) we get that $b_2 > 0$, and if $a_2^2 - b_2 \geqslant 0$, then $a_2 \geqslant \sqrt{b_2} > 0$. This implies (16_1). Similarly, it follows from (i) that, if $a_1^2 - b_2 \geqslant 0$, then $a_1 \leqslant -\sqrt{b_2} < 0$. This implies (16_3). Summing the above, the following lemma may be stated.

LEMMA 1. *Assume the inequality (i). Then the solutions of (6) are oscillatory, if and only if one of the following cases holds:*

a) $\lambda_i(a_j, b_1)$ *are imaginary numbers and* ϱ *is arbitrary* $(j = 1, 2)$,

b) $\lambda_1(a_2, b_1)$ *is real-valued, non-positive and* $\varrho < \lambda_1(a_2, b_1)$,

c) $\lambda_1(a_2, b_1)$ *is a real positive or an imaginary number,* $\lambda_1(a_1, b_1)$ *is real-valued and* $\varrho < \lambda_1(a_1, b_1)$.

Analogous reasoning leads to the following lemma.

LEMMA 2. *Assume the inequality (i). Then the solutions of (7) are oscillatory if and only if one of the following cases holds:*

A) $\lambda_i(a_j, b_1)$ *are imaginary numbers and* ϱ *is arbitrary* $(j = 1, 2)$,

B) $\lambda_2(a_1, b_1)$ *is a real-valued, non-negative number and* $\varrho > \lambda_2(a_1, b_1)$,

C) $\lambda_2(a_1, b_1)$ *is a real negative number or an imaginary number,* $\lambda_2(a_2, b_1)$ *is real-valued and* $\varrho > \lambda_2(a_2, b_1)$.

5. Assume in the following that the solutions of (6) and of (7) are oscillatory. Let the functions $W_i(u, v)$ $(i = 1, 2)$ be defined as follows

$$W_i(0, v) = v \quad \text{for} \quad v > 0, \tag{17}$$

$$W_i(u(t), v(t)) = \text{const.} \quad \text{for} \quad T_1 \leqslant t \leqslant T_2, \tag{18}$$

where $u(t), v(t)$ is a solution of (6), if $i = 1$, and of (7), if $i = 2$, and $u(0) = 0, v(0) = v_0$ $(v_0 > 0)$. T_1 and T_2 denote the greatest negative zero and the smallest positive zero of the function $v(t) - \varrho u(t)$, respectively *). Functions $W_i(u, v)$ are by (17) and (18) defined uniquely in the set $v - \varrho u \geqslant 0$. In order to define $W_i(u, v)$ in the whole plane (u, v), we put

$$W_i(u, v) = W_i(-u, -v), \quad \text{if} \quad v - \varrho u < 0 \quad (i = 1, 2). \tag{19}$$

The functions $W_i(u, v)$ are of class C^1 everywhere with the exception of the line $v = \varrho u$. They are continuous on this line if and only if

$$W_i(u, \varrho u) = W_i(-u, -\varrho u), \quad (i = 1, 2), \tag{20}$$

i. e. if the solutions of (6) or of (7) are periodic.

Due to (17), (18) and (19) we have the following relations

$$(\partial W_i/\partial u)^2 + (\partial W_i/\partial v)^2 > 0, \tag{21}$$

$$(-\varrho u + v)\,\partial W_i/\partial u + \sigma_i(u, v)\,\partial W_i/\partial v = 0, \quad \text{if} \quad v \neq \varrho u \ (i = 1, 2). \tag{22}$$

*) Note that T_1 and T_2 do not depend on v_0.

It follows from (21), (22) and (17) that

(23) $\partial W_i/\partial v > 0$ for $v > \varrho u$ and $\partial W_i/\partial v < 0$ for $v < \varrho u$ $(i = 1, 2)$.

Finally, by (11), (18) and (19), we get that

(24) $W_i(cu, cv) = |c| W_i(u, v)$, where c is const., $(i = 1, 2)$.

6. LEMMA 3. *Consider one of the cases a)-c) of Lemma* 1, *and assume that* $a(t)$ *and* $b(t)$, *in* (5), *satisfy* (2). *Let* $u(t), v(t)$ *be an arbitrary solution of* (5). *Then the function* $\varphi(t) = W_1(u(t), v(t))$ *is non-decreasing on each interval on which we have* $v(t) - \varrho u(t) \neq 0$.

Moreover, if we assume that

(25) $$W_1(u, \varrho u) \leqslant W_1(-u, -\varrho u) \quad (u > 0),$$

then $\varphi(t)$ *is non-decreasing for all* t.

Proof. Notice that $v(t) - \varrho u(t)$ may be equal to zero on a set of isolated points. Hence $\varphi'(t)$ exists everywhere with the exception at most of a denumerable set of points. Since

$$\varphi'(t) = \big(-\varrho u(t) + v(t)\big)\partial W_i/\partial u - \big((2a(t) + \varrho)v(t) + b(t)u(t)\big)\partial W_i/\partial v$$

we get by (12), (22) and (23) that

(26) $$\varphi'(t) = -\big(\sigma_1(u(t), v(t)) + (2a(t) + \varrho)v(t) + b(t)u(t)\big)\partial W_1/\partial v \geqslant 0.$$

The inequality (26) proves the first part of Lemma 3. Suppose now that for $t = t_*$ we have $v(t_*) - \varrho u(t_*) = 0$. By (25) we get the inequality $\lim_{t \to t_* - 0} \varphi(t) \leqslant \lim_{t \to t_* + 0} \varphi(t)$. This inequality commonly with (26) completes the proof of Lemma 3.

Similarly one can prove the following lemma concerning $W_2(u, v)$.

LEMMA 4. *Consider one of the cases A)-C) of Lemma* 2, *and assume that* $a(t)$ *and* $b(t)$, *in* (5), *satisfy* (2). *Let* $u(t), v(t)$ *be an arbitrary nontrivial solution of* (5). *Then the function* $\psi(t) = W_2(u(t), v(t))$ *is non-increasing on each interval on which we have* $v(t) - \varrho u(t) \neq 0$. *Moreover, if we assume that*

(27) $$W_2(u, \varrho u) \geqslant W_2(-u, -\varrho u) \quad (u > 0),$$

then $\psi(t)$ *is non-increasing for all* t.

7. Note, that, due to (24), the inequalities (25) and (27) are equivalent to the following ones, respectively:

(28) $W_1(1, \varrho) \leqslant W_1(-1, -\varrho)$ and $W_2(1, \varrho) \geqslant W_2(-1, -\varrho)$.

Let $u_*(t), v_*(t)$ be a solution of (14) such that $u(0) = 0, v(0) = 1$. Denote by t_0 the smallest positive number such that $v_*(t_0) - \varrho u_*(t_0) = 0$, if $\varrho > 0$ and $v_*(t_0) = 0$, if $\varrho \leqslant 0$; similarly denote by t_1 the greatest negative number such that $v_*(t_1) - \varrho u_*(t_1) = 0$, if $\varrho < 0$ and $v_*(t_1) = 0$, if $\varrho \geqslant 0$. We put

$$A(a, b, \varrho) = u_*(t_0) \quad \text{and} \quad B(a, b, \varrho) = u_*(t_1).$$

Let $u_{**}(t), v_{**}(t)$ be a solution of (14) such that $u_{**}(0)=1$, $v_{**}(0)=0$. Denote by t_2 the greatest negative number, if $\varrho>0$, and the smallest positive number, if $\varrho<0$, such that $v_{**}(t_2)-\varrho u_{**}(t_2)=0$. We put $C(a,b,\varrho)=u_{**}(t_2)$.

We give below the explicit formulae for $A(a,b,\varrho)$, $B(a,b,\varrho)$ and $C(a,b,\varrho)$, which may be obtained by elementary calculations omitted here. In the case $a^2-b<0$ we have

$$A(a,b,\varrho)=\Phi(a,b,\varrho,0,\alpha(\varrho)), \qquad B(a,b,\varrho)=\Phi(a,b,\varrho,1,1-\alpha(\varrho))$$

$$C(a,b,\varrho)=\frac{\sqrt{b}}{\sqrt{\varrho^2+2a\varrho+b}}\exp\left(\frac{-a-\varrho}{\sqrt{b-a^2}}\left(-\alpha(\varrho)\pi+\operatorname{arc\,ctg}\frac{-b-a\varrho}{\varrho\sqrt{b-a^2}}\right)\right),$$

where $\Phi(a,b,\varrho,\eta,\xi)=\dfrac{1-2\eta}{\sqrt{\xi(\varrho^2+2a\varrho)+b}}\exp\left(\dfrac{-a-\varrho}{\sqrt{b-a^2}}\left(-\eta\pi+\operatorname{arc\,ctg}\dfrac{\xi\varrho+a}{\sqrt{b-a^2}}\right)\right)$ and $\alpha(\varrho)=1$ for $\varrho>0$, $\alpha(\varrho)=0$ for $\varrho\leqslant 0$. In the case $a^2-b\geqslant 0$ we have $A(a,b,\varrho)=\psi(a,b,\varrho,\beta(\varrho))$, $B(a,b,\varrho)=-\psi(a,b,\varrho,1-\beta(\varrho))$, where

$$\psi(a,b,\varrho,\xi)=\left|a-\sqrt{a^2-b}+\xi\varrho\right|^{\frac{a-\sqrt{a^2-b}+\varrho}{2\sqrt{a^2-b}}}\left|a+\sqrt{a^2-b}+\xi\varrho\right|^{\frac{-a-\sqrt{a^2-b}-\varrho}{2\sqrt{a^2-b}}},$$

$\beta(\varrho)=1$ for $\varrho>0$ and $\beta(\varrho)=0$ for $\varrho<0$.

$$C(a,b,\varrho)=\left(\frac{a-\sqrt{a^2-b}}{a-\sqrt{a^2-b}+\varrho}\right)^{\frac{-a+\sqrt{a^2-b}-\varrho}{2\sqrt{a^2-b}}}\left(\frac{a+\sqrt{a^2-b}}{a+\sqrt{a^2-b}+\varrho}\right)^{\frac{a+\sqrt{a^2-b}+\varrho}{2\sqrt{a^2-b}}},$$

where $\varrho>-a+\sqrt{a^2-b}$, if $-a+\sqrt{a^2-b}<0$ and $\varrho<-a-\sqrt{a^2-b}$, if $-a-\sqrt{a^2-b}>0$.

The following relations may be easily verified

$$\lim_{\varrho\to+\infty}A(a,b,\varrho)=\lim_{\varrho\to-\infty}B(a,b,\varrho)=0, \tag{29}$$

$$\lim_{\varrho\to-\infty}A(a,b,\varrho)=\lim_{\varrho\to+\infty}-B(a,b,\varrho)=\lim_{|\varrho|\to\infty}C(a,b,\varrho)=+\infty. \tag{30}$$

Using the above defined functions we get by (17), (18) and (24) the following formulae:

$$W_1(1,\varrho)A(a_2,b_2,\varrho)=1, \qquad W_1(-1,-\varrho)B(a_2,b_1,\varrho)C(a_1,b_1,\varrho)=-1 \quad \text{for} \quad \varrho>0, \tag{31}$$

$$W_1(1,\varrho)A(a_2,b_2,\varrho)C(a_1,b_2,\varrho)=1, \qquad W_1(-1,-\varrho)B(a_2,b_1,\varrho)=-1 \quad \text{for} \quad \varrho\leqslant 0, \tag{32}$$

$$W_2(1,\varrho)A(a_1,b_1,\varrho)=1, \qquad W_2(-1,-\varrho)B(a_1,b_2,\varrho)C(a_2,b_2,\varrho)=-1 \quad \text{for} \quad \varrho>0, \tag{33}$$

$$W_2(1,\varrho)A(a_1,b_1,\varrho)C(a_2,b_1,\varrho)=1, \qquad W_2(-1,-\varrho)B(a_1,b_2,\varrho)=-1 \quad \text{for} \quad \varrho\leqslant 0. \tag{34}$$

Remark 2. It follows from (29)-(34) that inequalities (25) or (27) hold if ϱ is sufficiently small or ϱ is sufficiently great, respectively. Further, if the equation

$$W_1(1, \varrho) = W_1(-1, -\varrho) \tag{35}$$

has no root, then $W_1(1, \varrho)$ is defined only on the half-line $\varrho < \varrho_* < +\infty$ and (25) holds for $\varrho < \varrho_*$. Similarly, if the equation

$$W_2(1, \varrho) = W_2(-1, -\varrho) \tag{36}$$

has no solution, then $W_2(1, \varrho)$ is defined only on the half-line $\varrho > \varrho_{**} > -\infty$ and (27) holds for $\varrho > \varrho_{**}$. Due to Lemma 1 or 2, the latter may occur if and only if one of the numbers $\lambda_i(a_j, b_1)$, $j = 1, 2$, is real-valued. The constants ϱ_* and ϱ_{**} have the following values:

$$\varrho_* = \lambda_1(a_2, b_1)\,, \text{ if } \lambda_1(a_2, b_1) \leqslant 0 \text{ or } \varrho_* = \lambda_1(a_1, b_1) \quad \text{in other cases,} \tag{37}$$

$$\varrho_{**} = \lambda_2(a_1, b_1)\,, \text{ if } \lambda_2(a_1, b_1) \geqslant 0 \text{ or } \varrho_{**} = \lambda_2(a_2, b_1) \quad \text{in other cases.} \tag{38}$$

8. THEOREM 1. *Let us consider Eq.* (1). *Assume that* $a(t), b(t)$, *in* (1), *satisfy* (2), *and that condition* (*i*) *holds. Further, denote by* ϱ_1 *the smallest root of* (35), *if such a root exists, and put* ϱ_1 *equal to* ϱ_*, *if* (35) *has no solution. By* ϱ_2 *we denote the greatest root of* (36) *), *if such a root exists, and if* (36) *has no solution, then we put* ϱ_2 *equal to* ϱ_{**}. *The values of* ϱ_* *and* ϱ_{**} *are given by* (37) *and* (38). *Under these assumptions*:

1° *For each nontrivial solution* $x(t)$ *of* (1) *the following inequalities hold*

$$\limsup_{t\to+\infty} |x(t)| \exp(-\varrho t) > 0\,, \quad \limsup_{t\to+\infty} |x'(t)| \exp(-\varrho t) > 0 \quad \text{for } \varrho < \varrho_1 \tag{39}$$

$$\limsup_{t\to+\infty} |x(t)| \exp(-\varrho t) < +\infty\,, \quad \limsup_{t\to+\infty} |x'(t)| \exp(-\varrho t) < +\infty \quad \text{for } \varrho > \varrho_2. \tag{40}$$

2° *If* ϱ_1 (*or* ϱ_2) *is defined as a root of* (35) (*or* (36)), *then* (39) (*or* (40)) *holds also for* $\varrho = \varrho_1$ (*or for* $\varrho = \varrho_2$).

3° *There exist functions* $\bar{a}(t), \bar{b}(t)$ (*or* $\bar{\bar{a}}(t), \bar{\bar{b}}(t)$) *satisfying* (2) *such that the equation*

$$x'' + \bar{a}(t)x' + \bar{b}(t)x = 0 \quad (\text{or } x'' + \bar{\bar{a}}(t)x' + \bar{\bar{b}}(t)x = 0) \tag{41}$$

admits a solution $\bar{x}(t)$ (*or* $\bar{\bar{x}}(t)$) *for which* (39) (*or* (40)) *does not hold, if* $\varrho > \varrho_1$ (*or* $\varrho < \varrho_2$).

Proof. We restrict ourselves to those parts of Theorem 1 which concern ϱ_1. The proof of the remaining assertions is analogous.

Due to Remark 2, the inequality (25) holds for $\varrho < \varrho_1$. Hence, owing to Lemma 3, we get for $\varrho < \varrho_1$ and for an arbitrary nontrivial solution $u(t), v(t)$ of (5) the inequality

$$W_1(u(t), v(t)) \geqslant C > 0\,, \quad (t > t_0)\,. \tag{42}$$

*) In fact (35), as well as (36), have one root at most.

It follows from (42) that $\limsup_{t\to+\infty}|u(t)|>0$ and $\limsup_{t\to+\infty}|v(t)|>0$. The latter inequalities, due to Remark 1, complete the proof of 1°.

If ϱ_1 is a root of (35), then (42) holds also for $\varrho=\varrho_1$. This proves 2°.

Suppose now that $\varrho_1=\varrho_*$. In order to prove 3° in this case it suffices to put $\bar{b}(t)=b_1$ and $\bar{a}(t)=a_1$ or $\bar{a}(t)=a_2$, if $\varrho_*=\lambda_1(a_1,b_1)$ or $\varrho_*=\lambda_1(a_2,b_1)$, respectively.

Assume now that ϱ_1 is defined as a root of (35). Then every solution of (6) for $\varrho=\varrho_1$ is periodic. Let $\bar{u}(t)$, $\bar{v}(t)$ be a nontrivial solution of (6). Suppose that $\bar{u}(0)=0$ and $\bar{v}(0)>0$. Let numbers t_1,t_2,t_3 be so chosen that $\bar{v}(t_1)-\varrho_1\bar{u}(t_1)=0$, $\bar{v}(t_2)=0$, $\bar{u}(t_3)=0$ and $\bar{u}(t)>0$ for $0<t<t_3$, where $0<t_1<t_3$, $0<t_2<t_3$ and $t_1<t_2$, if $\varrho>0$, $t_2\leqslant t_1$, if $\varrho\leqslant 0$. Define $\bar{a}(t)$, $\bar{b}(t)$, in (41), as follows: $\bar{a}(t)=a_2$ for $0\leqslant t<\min(t_1,t_2)$ and for $\max(t_2,t_1)\leqslant t<t_3$, $\bar{a}(t)=a_1$ for $\min(t_1,t_2)\leqslant t<\max(t_1,t_2)$; $\bar{b}(t)=b_2$ for $0\leqslant t<t_2$ and $\bar{b}(t)=b_1$ for $t_2\leqslant t<t_3$. Besides, let us define $\bar{a}(t)$, $\bar{b}(t)$ as periodic functions with period equal to t_3. It may be easily shown that $\bar{x}(t)=\bar{u}(t)\exp(\varrho_1 t)$ is a nontrivial solution of (41) and that (39) does not hold for $\bar{x}(t)$, if $\varrho>\varrho_1$. This proves 3° and at the same time completes the proof of Theorem 1.

Remark 3. Our results remain valid without any changes if, instead of (1), we consider the non-linear equation

$$x''+f(t,x,x')x'+g(t,x,x')x=0\,,$$

and if, instead of (2), we adopt the inequalities

$$a_1\leqslant f(t,x,y)\leqslant a_2\,,\qquad b_1\leqslant g(t,x,y)\leqslant b_2\,,\qquad \text{for all } (t,x,y)\,.$$

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MATHÉMATIQUE

Sur la stabilité des solutions périodiques et presque-périodiques de l'équation différentielle

$$x'' + F(x') + g(x) = p(t)$$

par

Z. OPIAL

Présenté par T. WAŻEWSKI le 6 juin 1959

1. Envisageons l'équation différentielle du second ordre

$$x'' + F(x') + g(x) = p(t)\,, \tag{1}$$

équivalente au système d'équations

$$x' = y\,,\quad y' = -F(y) - g(x) + p(t)\,. \tag{2}$$

Supposons que les fonctions $g(x)$, $F(y)$ et $p(t)$ soient continues et que l'on ait de plus

$$\lim_{|x|\to\infty} g(x)\operatorname{sgn} x = +\infty\,,\qquad \lim_{|y|\to\infty} F(y)\operatorname{sgn} y = +\infty\,,\qquad |p(t)| \leqslant P\,, \tag{3}$$

P étant une constante.

G. E. H. Reuter [6] a démontré que dans ces hypothèses il existe un ensemble K du plan (x, y), limité par une courbe simple de Jordan, et jouissant des propriétés suivantes:

1° pour toute solution $(x(t), y(t))$ du système (2) il existe un t_0 tel que $(x(t_0), y(t_0)) \in K$;

2° si pour un t_0 $(x(t_0), y(t_0)) \in K$, on a aussi $(x(t), y(t)) \in K$ pour tout $t \geqslant t_0$.

Il existe donc deux constantes B_1 et B_2 telles, que pour toute solution $x(t)$ de l'équation (1) on a

$$|x(t)| \leqslant B_1 \quad \text{et} \quad |x'(t)| \leqslant B_2\,, \tag{4}$$

pourvu que t soit suffisamment grand. Si, en particulier, la fonction $p(t)$ est périodique de période ω, il existe au moins une solution périodique de l'équation (1) ayant la même période.

D'autres conditions qui assurent les mêmes propriétés des solutions de l'équation (1) ont été établies par R. Reissig [4]. L'équation (1) a été

étudiée par A. Ascari [1] dans le cas où $g(x) \equiv x$ (cf. aussi [7], p. 499 et suiv.). Il a démontré les inégalités (4) dans l'hypothèse

$$\liminf_{y\to+\infty} F(y) > P \quad \text{et} \quad \limsup_{y\to-\infty} F(y) < -P\,. \tag{5}$$

Si, en particulier, la fonction $p(t)$ est périodique de période ω, les inégalités (5) sont bien suffisantes à assurer l'existence d'au moins une solution périodique de l'équation

$$x''+F(x')+x = p(t)\,. \tag{1*}$$

Si, de plus, la fonction $F(y)$ est croissante, cette solution périodique est unique et stable c'est-à-dire toute autre solution de l'équation (1*) tend vers cette solution périodique lorsque $t\to+\infty$ (cf. aussi R. Reissig [3]).

Dans la présente note j'établis un théorème (Théorème 5) de l'unicité et de stabilité de la solution périodique de l'équation générale (1). La méthode dont je me sers à cet effet est pareille à celle exposée dans les travaux [2] et [8]. Dans le dernier paragraphe du présent travail je démontre l'existence et la stabilité d'une solution presque-périodique de l'équation (1) dans l'hypothèse que la fonction $p(t)$ est presque-périodique. On obtient ainsi un théorème analogue au théorème établi par G. E. H. Reuter [5] pour l'équation $x''+f(x)x'+g(x) = p(t)$.

2. Nous commencerons par établir un théorème qui nous permettra de comparer les solutions de deux équations du type (1). Envisageons à cet effet les équations

$$x''+F(x')+g(x) = p_1(t)\,, \tag{1'}$$

$$x''+F(x')+g(x) = p_2(t)\,, \tag{1''}$$

dont chacune est équivalente respectivement au système

$$x' = y\,, \quad y' = -F(y)-g(x)+p_1(t)\,, \tag{2'}$$

$$x' = y\,, \quad y' = -F(y)-g(x)+p_2(t)\,. \tag{2''}$$

Théorème 1. *Supposons que les fonctions $g(x), F(y), p_1(t)$ et $p_2(t)$ satisfont aux conditions* (3) *et aux inégalités*

$$f(y) = F'(y) > 0 \quad \textit{pour} \quad |y| \leqslant B_2\,, \tag{6}$$

$$g'(x) > 0 \quad \textit{pour} \quad |x| \leqslant B_1\,, \tag{7}$$

$$2 \min_{|y|\leqslant B_2} f(y)/|y| > \max_{|x|\leqslant B_1} |g''(x)|/g'(x)\,, \tag{8}$$

les constantes B_1 et B_2 étant les mêmes que dans les inégalités (4). *Posons $\Delta p(t) = p_2(t)-p_1(t)$ et supposons que l'on ait*

$$|\Delta p(t)| \leqslant \varepsilon \quad (-\infty < t < +\infty)\,. \tag{9}$$

Ceci étant admis, il existe une constante $M > 0$ *indépendante de* ε *telle que pour deux solutions arbitraires* $x_1(t)$ *et* $x_2(t)$ *des équations* (1′) *et* (1″) *respectivement, on a les inégalités*

$$\limsup_{t\to+\infty} |x_1(t)-x_2(t)| \leqslant M\varepsilon\,, \qquad \limsup_{t\to+\infty} |x_1'(t)-x_2'(t)| \leqslant M\varepsilon\,. \tag{10}$$

Démonstration. Soient $(x_1(t), y_1(t))$ et $(x_2(t), y_2(t))$ deux solutions arbitraires des systèmes (2′) et (2″) respectivement. L'ensemble K dont il était question dans le théorème de G. Reuter cité au n^0 1 ne dépend que des fonctions $g(x), F(y)$ et de la constante P, il est donc le même pour les équations (1′) et (1″). Il existe donc un t_0 tel que l'on a $(x_1(t), y_1(t)) \in K$ et $(x_2(t), y_2(t)) \in K$ pour $t \geqslant t_0$. Joignons le point $(x_1(t_0), y_1(t_0))$ au point $(x_2(t_0), y_2(t_0))$ par une courbe régulière C qui appartient entièrement à l'ensemble K:

$$(C) \qquad x = \varphi(u)\,, \quad y = \psi(u) \quad 0 \leqslant u \leqslant 1$$

où les fonctions $\varphi(u)$ et $\psi(u)$ sont de classe C^1 et telles que

$$\varphi(0) = x_1(t_0)\,, \quad \psi(0) = y_1(t_0)\,, \quad \varphi(1) = x_2(t_0) \quad \text{et} \quad \psi(1) = y_2(t_0)\,.$$

Désignons par $(x(t, u), y(t, u))$ cette solution du système auxiliaire

$$x' = y\,, \quad y' = -F(y) - g(x) + p_1(t) + u\Delta p(t) \tag{11}$$

qui, pour $t = t_0$, passe par le point $(\varphi(u), \psi(u))$ de la courbe C. On a ainsi

$$x_1(t) = x(t, 0)\,, \quad y_1(t) = y(t, 0)\,, \quad x_2(t) = x(t, 1)\,, \quad y_2(t) = y(t, 1)\,.$$

Désignons par $C(t)$ $(t \geqslant t_0)$ la courbe

$$(C(t)) \qquad x = x(t, u)\,, \quad y = y(t, u) \quad 0 \leqslant u \leqslant 1\,.$$

En vertu de la seconde propriété de l'ensemble K pour tout $t \geqslant t_0$ la courbe $C(t)$ appartient entièrement à K. On a donc

$$|x(t, u)| \leqslant B_1\,, \quad |y(t, u)| \leqslant B_2 \quad t \geqslant t_0\,,\ 0 \leqslant u \leqslant 1\,. \tag{12}$$

Il est facile d'évaluer la distance des points $(x_1(t), y_1(t))$ et $(x_2(t), y_2(t))$ à l'aide de la longueur de la courbe $C(t)$. On a notamment

$$|x_1(t)-x_2(t)| + |y_1(t)-y_2(t)| \leqslant 2\int_0^1 \left\{\left(\frac{\partial x(t, u)}{\partial u}\right)^2 + \left(\frac{\partial y(t, u)}{\partial u}\right)^2\right\}^{1/2} du\,.$$

Pour démontrer les inégalités (10) il suffit donc de démontrer que les fonctions $v_1(t, u) = \partial x(t, u)/\partial u$, $v_2(t, u) = \partial y(t, u)/\partial u$ sont bornées et qu'il existe une constante N telle que

$$\limsup_{t\to+\infty} |v_1(t, u)| \leqslant N\varepsilon\,, \qquad \limsup_{t\to+\infty} |v_2(t, u)| \leqslant N\varepsilon\,. \tag{13}$$

Les fonctions $v_1(t, u)$, $v_2(t, u)$ satisfont au système d'équations différentielles

$$v_1' = v_2\,, \qquad v_2' = -f\big(y(t, u)\big)v_2 - g'\big(x(t, u)\big)v_1 + \Delta p(t)\,. \tag{14}$$

Posons pour tout $t \geqslant t_0$ et $0 \leqslant u \leqslant 1$:

$$G(t, u) = v_1^2(t, u) + \frac{v_2^2(t, u)}{g'\big(x(t, u)\big)} + \eta v_1(t, u)\, v_2(t, u)\,,$$

où η est une constante positive qui va être déterminée dans la suite. Remarquons que pour η suffisamment petit la fonction $G(t, u)$ est une forme quadratique en v_1 et v_2, définie positive. On a de plus, en vertu de (14):

$$\frac{\partial G(t, u)}{\partial t} = -\eta g'(x) v_1^2 - \eta f(y) v_1 v_2 - \left(\frac{2f(y)g'(x) + g''(x)y}{g'^2(x)} - \eta\right) v_2^2 + \left(\frac{2v_2}{g'(x)} + \eta v_1\right)\Delta p(t) \tag{15}$$

où nous écrivons v_1, v_2, x et y au lieu de $v_1(t, u)$ etc. En vertu des inégalités (8) et (12) on peut choisir η de manière que la forme quadratique dans le deuxième membre de la relation (15) soit définie négative. On a, par conséquent, pour les constantes η et α suffisamment petites:

$$\frac{\partial G(t, u)}{\partial t} \leqslant -\alpha G(t, u) + \left(\frac{2}{g'(x)}|v_2| + \eta|v_1|\right)|\Delta p(t)|\,.$$

En vertu de l'inégalité (9) il en vient que pour un β suffisamment grand

$$\frac{\partial G(t, u)}{\partial t} \leqslant -\alpha G(t, u) + \beta\varepsilon\sqrt{G(t, u)} \qquad (t \geqslant t_0,\ 0 \leqslant u \leqslant 1)\,. \tag{16}$$

Il en résulte que $\limsup_{t\to+\infty} G(t, u) \leqslant \beta^2\varepsilon^2/\alpha^2$. Comme $G(t, u)$ est une forme quadratique définie positive, il en vient que les inégalités (13) sont satisfaites pour un N convenablement choisi.

3. De la démonstration du Théorème 1 il vient immédiatement que ce théorème peut être formulé comme suit:

THÉORÈME 2. *Supposons que les fonctions $g(x)$, $F(y)$, $p_1(t)$ et $p_2(t)$ satisfont aux hypothèses du Théorème* 1. *Il existe un M indépendant de ε et un $T(\varepsilon)$ tels que pour tout couple $x_1(t)$, $x_2(t)$ des solutions des équations* (1′) *et* (1″) *respectivement, pour lesquelles*

$$\big(x_1(t_0)\,,\ x_1'(t_0)\big) \in K\,, \qquad \big(x_2(t_0)\,,\ x_2'(t_0)\big) \in K\,,$$

on a pour tout $t \geqslant t_0 + T(\varepsilon)$:

$$|x_1(t) - x_2(t)| \leqslant M\varepsilon\,, \qquad |x_1'(t) - x_2'(t)| \leqslant M\varepsilon\,. \tag{17}$$

En effet, la frontière de l'ensemble K est une courbe de classe C^1, sauf en un nombre fini de points. Tout point P_1 de cet ensemble se laisse donc joindre à tout autre point P_2 appartenant à K par une courbe régulière $C \subset K$, dont la longueur ne dépasse pas un nombre convenablement choisi. Par conséquent, les valeurs de la fonction $G(t_0, u)$ $(0 \leqslant u \leqslant 1)$ sont bornées par une constante indépendante du choix des points P_1 et P_2. Mais de l'inégalité différentielle (16) il vient que l'on doit avoir

$$0 \leqslant G(t, u) \leqslant 4\beta^2\varepsilon^2/\alpha^2 \qquad (0 \leqslant u \leqslant 1)$$

pour tout $t \geqslant t_0 + T(\varepsilon)$, où $T(\varepsilon)$ ne dépend ni du choix des points P_1, P_2, ni de t_0.

4. Du Théorème 2 on obtient immédiatement le théorème suivant:

THÉORÈME 3. *Si les fonctions $g(x)$, $F(y)$, $p_1(t)$ et $p_2(t)$ satisfont aux hypothèses du Théorème* 1 *et $x_1(t), x_2(t)$ sont deux solutions des équations* (1′) *et* (1″) *respectivement, telles que*

$$\big(x_1(t)\,,\ x_1'(t)\big) \in K \qquad et \qquad \big(x_2(t)\,,\ x_2'(t)\big) \in K \qquad (-\infty < t < +\infty)\,,$$

dans tout l'intervalle $(-\infty, +\infty)$ on a les inégalités (17).

5. Dans le cas où $p_1(t) \equiv p_2(t)$ on déduit du Théorème 1 le suivant

THÉORÈME 4. *Si les fonctions $g(x)$, $F(y)$ et $p(t)$ satisfont aux hypothèses du Théorème* 1, *pour tout couple $x_1(t), x_2(t)$ des solutions de l'équation* (1) *on a*

$$\lim_{t\to+\infty} |x_1(t) - x_2(t)| = 0\,, \qquad \lim_{t\to+\infty} |x_1'(t) - x_2'(t)| = 0\,. \tag{18}$$

Sous l'hypothèse que la fonction $p(t)$ est périodique on en obtient le théorème suivant:

THÉORÈME 5. *Si les fonctions $g(x)$, $F(y)$ et $p(t)$ satisfont aux hypothèses du Théorème* 1 *et la fonction $p(t)$ est périodique de période ω, l'équation* (1) *admet une et une seule solution périodique $x_1(t)$ de période ω. Pour toute autre solution $x_2(t)$ de cette équation on a les relations* (18).

De la propriété 2° de l'ensemble K (cf. n° 1) il résulte qu'il existe au moins une solution $x(t)$ de l'équation (1) telle que $\big(x(t), x'(t)\big) \in K$ dans tout l'intervalle $(-\infty, +\infty)$. Mais du Théorème 3 il vient que l'équation (1) admet au plus une telle solution. On a donc le suivant

THÉORÈME 6. *Si les fonctions $g(x)$, $F(y)$ et $p(t)$ satisfont aux hypothèses du Théorème* 1, *l'équation* (1) *admet une et une seule solution $x(t)$, pour laquelle $\big(x(t), x'(t)\big) \in K$ dans tout l'intervalle $(-\infty, +\infty)$.*

6. Supposons maintenant que la fonction $p(t)$ soit presque-périodique. Il est facile de démontrer que dans cette hypothèse l'équation (1) admet une solution presque-périodique stable.

Rappelons d'abord qu'une fonction continue $p(t)$ est appelée presque-périodique si, à tout $\varepsilon > 0$, on peut faire correspondre un $L(\varepsilon) > 0$ tel, que tout intervalle de longueur $L(\varepsilon)$ contient au moins une presque-période relative à ε, c'est-à-dire un τ tel, que

$$|p(t)-p(t+\tau)| \leqslant \varepsilon \quad (-\infty < t < +\infty).$$

THÉORÈME 7. *Si les fonctions $g(x)$, $F(y)$ et $p(t)$ satisfont aux hypothèses du Théorème* 1 *et la fonction $p(t)$ est presque-périodique, l'équation* (1) *admet une et une seule solution presque-périodique.*

Démonstration. Désignons par $\alpha(t)$ cette solution de l'équation (1) pour laquelle $(\alpha(t), \alpha'(t)) \in K$ dans tout l'intervalle $(-\infty, +\infty)$ (cf. théorème 6). Nous allons montrer que $\alpha(t)$ et $\alpha'(t)$ sont des fonctions presque-périodiques.

Soit τ une presque-période de la fonction $p(t)$, relative à ε/M (M — constante intervenant dans l'énoncé du théorème 2). La fonction $\alpha(t+\tau)$ est évidemment une solution de l'équation (1'') où l'on a posé $p_2(t) \equiv p(t+\tau)$. On a de plus $(\alpha(t+\tau), \alpha'(t+\tau)) \in K$ pour tout t. Du théorème 3 (où l'on admet $p_1(t) \equiv p(t)$ et $p_2(t) \equiv p(t+\tau)$) il vient que dans tout l'intervalle $(-\infty, +\infty)$ les fonctions $\alpha(t)$ et $\alpha'(t)$ satisfont aux inégalités

$$|\alpha(t+\tau)-\alpha(t)| \leqslant \varepsilon, \qquad |\alpha'(t)-\alpha'(t+\tau)| \leqslant \varepsilon,$$

c'est-à-dire toute presque-période de la fonction $p(t)$ relative à ε/M est une presque-période des fonctions $\alpha(t)$ et $\alpha'(t)$ relative à ε. Le théorème 7 se trouve ainsi démontré.

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MATHEMATICS

Algebras Independently Generated by Every n Elements

by

S. ŚWIERCZKOWSKI

Presented by E. MARCZEWSKI on June 17, 1959

We use the notion of independence defined by E. Marczewski in [2]. Given an algebra A, a set $S \subset A$ is said to generate A if every element of the algebra is the result of some algebraic operation on a finite number of elements of S. An algebra A is called *independently generated by every n elements* if the power of A is at least n, each set $S \subset A$ of power n generates A and moreover all elements of such a set are independent. We call an algebra A *trivial* if there are in A only the identity operations:

$$(*) \qquad e_{i,j}(x_1, \ldots, x_i, \ldots, x_j) \equiv x_i; \qquad i = 1, \ldots, j;\ j = 1, 2, \ldots$$

The following results are announced: *)

THEOREM 1. *There is only one non-trivial algebra which is independently generated by every 3 elements. This algebra has 4 elements and all operations are generated by the operation $f(x, y, z)$ which is defined by the conditions*

$$(\alpha) \qquad f(x, y, z) \neq x, y, z \quad \text{for different } x, y, z$$

$$(\beta) \qquad f(x, x, y) = f(x, y, x) = f(y, x, x) = y \quad \text{for arbitrary } x, y\,.$$

THEOREM 2. *If an algebra is independently generated by every n elements and $n \geqslant 4$, then it is a trivial algebra with n elements.*

THEOREM 3. *Every infinite cardinal m is the power of a non-trivial algebra which is independently generated by every 2 elements. A finite cardinal m is the number of elements of such an algebra if and only if m is a power of a prime.*

If in Theorem 2 the algebra has exactly n elements, then the thesis remains valid when $n \geqslant 4$ is replaced by $n \neq 2$. This last condition is indispensable, as shown by a counter example due to E. Marczewski.

*) We expect to publish the proofs in Fundamenta Mathematicae.

Given a field F, the operations

$$f(x_1, \ldots, x_j) = \Sigma_1^j a_i x_i, \quad \text{where} \quad \Sigma_1^j a_i = 1;\ j = 1, 2, \ldots$$

define an algebra on the elements of F. A. M. Macbeath communicated to the present author his observation that this algebra is independently generated by every 2 elements. We make use of this in proving the sufficiency part of Theorem 3. The necessity part (for finite m) is proved by showing that the group of automorphisms of the algebra is a doubly transitive group of permutations of an m-element set, and then the result given in [1], section 105, is applied.

It might be expected that each algebra independently generated by every 2 elements is defined by a corresponding field F, as above (cf. [3]). A counter example (where the algebra has 9 elements) disproves this hypothesis.

There are algebras of arbitrary power which are independently generated by every element. We obtain such an algebra from each Abelian group, if the operations of adding to the variable a fixed element are the only admitted ones.

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ASTROPHYSICS

The Magnetogravitational Instability of a Medium in Nonuniform Rotation

by

A. G. PACHOLCZYK and J. S. STODÓŁKIEWICZ

Presented by W. IWANOWSKA on June 8, 1959

The gravitational instability of a medium in nonuniform rotation was examined by Bel and Schatzman [1] in the case of axisymmetrical perturbation. This problem, interesting from the point of view of the mechanism of formation of spiral arms, was solved in the case of absence of a magnetic field. Observations of interstellar polarization indicate the possibility of existence of a magnetic field in the interstellar space. It seems therefore interesting to examine the problem of instability in the case of the presence of a magnetic field. As follows from polarization data, the regular interstellar magnetic field is in the first approximation parallel to the direction of spiral arms. Therefore it seems plausible to assume, that in the gaseous protogalaxy, in which the spiral arms are formed by the instability against the axisymmetrical perturbation, the magnetic field has circular lines of force centered on the axis of rotation. The perturbation, symmetrical in respect to the axis of rotation of the system, can be considered as a circular wave, running in a direction perpendicular to the direction of the magnetic lines of force. We can therefore expect a stabilizing action of the magnetic field on the perturbation considered. Investigations of this stabilizing effect can give the upper limit of the magnetic field intensity in the model of the protogalaxy. This limiting intensity will be determined by the existence in our galaxy of spiral arms with observed dimensions.

We examine the plane problem of the magnetogravitational instability of the isothermal, non viscous medium of infinite electrical conductivity, participating in a nonuniform rotation. We assume, that the magnetic lines of force are circles centered on the axis of symmetry of the system. We further assume that the angular velocity vector is perpendicular to the plane of symmetry of the system.

Let ϱ denote the medium density, $\Omega(r)$ — the angular velocity of rotation, $u_r(r)$ and $u_\varphi(r)$ — the radial and tangential components of

the perturbation velocity in the plane of symmetry of the system, $\delta\psi(r)$ – the potential of the perturbed gravitational field, $H_\varphi(r)$ – the tangential component of the unperturbed magnetic field (we assume, that the radial component H_r vanishes), $h_r(r)$ and $h_\varphi(r)$ – the radial and tangential components of the magnetic field, due to the perturbation, V_s – the velocity of sound and G is the gravitational constant, therefore the linearized equations of the problem examined in the hydromagnetic approximation have the following forms

$$(1)\quad \varrho\dot{u}_r = -\delta p_{,r} - (4\pi r)^{-1} H_\varphi (h_\varphi r)_{,r} - (4\pi r)^{-1} h_\varphi (rH_\varphi)_{,r} + 2\varrho u_\varphi \Omega + \varrho\delta\psi_{,r} ,$$

$$(2)\quad \varrho\dot{u}_\varphi = (4\pi r)^{-1} h_r (rH_\varphi)_{,r} + \varrho u_r[-(\Omega r)_{,r} - \Omega] ,$$

$$(3)\quad \dot{h}_r = 0 ,$$

$$(4)\quad \dot{h}_\varphi = -H_\varphi u_{r,r} - u_r H_{\varphi,r} ,$$

$$(5)\quad \delta\dot{\varrho} + \varrho u_{r,r} + \varrho u_r r^{-1} + u_r \varrho_{,r} = 0 ,$$

$$(6)\quad r^{-1}(r\delta\psi_{,r})_{,r} + 4\pi G\delta\varrho = 0 ,$$

$$(7)\quad \delta p = V_s^2 \delta\varrho ,$$

with the condition

$$(8)\qquad (rh_r)_{,r} = 0 .$$

In the above equations, as well as in the whole paper vector components in the polar co-ordinate system are denoted by indices r, φ. Indices to the right of the comma denote the corresponding derivatives. Points above symbols make the time derivatives.

It follows from the divergence condition (8) and from (3) that we can put $h_r = 0$.

We consider solutions of (1)-(8) with the time-dependent factor of the form $\exp \omega t$. Thus we obtain

$$(9)\quad \varrho\omega u_r^* = -H_\varphi (4\pi r)^{-1} h_\varphi^* - H_\varphi (4\pi)^{-1} h_{\varphi,r}^* - (4\pi r)^{-1} h_\varphi^* (rH_\varphi)_{,r} + 2\varrho u_\varphi^* \Omega + \varrho\delta\psi_{,r}^* - \delta p_{,r}^* ,$$

$$(10)\quad \varrho\omega u_\varphi^* = \varrho u_r^*[-(\Omega r)_{,r} - \Omega] ,$$

$$(11)\quad \omega h_\varphi^* = -H_\varphi u_{r,r}^* - u_r^* H_{\varphi,r} ,$$

$$(12)\quad \omega\delta\varrho^* + \varrho u_{r,r}^* + \varrho u_r^* r^{-1} + u_r^* \varrho_{,r} = 0 ,$$

$$(13)\quad r^{-1}(r\delta\psi_{,r}^*)_{,r} + 4\pi G\delta\varrho^* = 0 ,$$

$$(14)\quad \delta p^* = V_s^2 \delta\varrho^* ,$$

and now all variables depend only on the r – co-ordinate. Substituting δp^* from (14), h_φ^* from (11) and u_φ^* from (10) into (9), we can write Eq. (9) in the form

$$(15)\quad \varrho\omega u_r^* = (4\pi\omega)^{-1} H_\varphi^2 u_{r,rr}^* + (4\pi\omega)^{-1}(2H_\varphi^2 r^{-1} + 3H_\varphi H_{\varphi,r}) u_{r,r}^* + (4\pi\omega)^{-1} \times$$
$$\times[(H_{\varphi,r})^2 + 2r^{-1} H_\varphi H_{\varphi,r} + H_{\varphi,rr} H_\varphi] u_r^* + 2\varrho\omega^{-1}\Omega[-(\Omega r)_{,r} - \Omega] u_r^* + \varrho\delta\psi_{,r}^* - V_s^2 \delta\varrho_{,r}^* .$$

Now we assume that $\varrho = \text{const}$. In further considerations we take the form of the magnetic field H_φ given by

$$H_\varphi = \text{const} \cdot r^{-1} . \tag{16}$$

This form follows from the unperturbed equations of motion in the case $\varrho = \text{const}$, if the gravitational force is balanced by the centrifugal force. The form of the instability criterion (29) obtained as the result of the present investigations is determined by assumption (16). In numerical calculations we use the approximate form (32) of the instability criterion, which is nearly independent of assumption (16). This approximate form gives the values of stabilizing magnetic field intensity which differ by less than $1^0/_0$ from the values obtained with the aid of criterion (29), so that the character of changes with the distance of the magnetic field H_φ has little influence on the stabilizing intensities, if the problem is considered locally at a great distance from the center of the system.

Eqs. (15) and (12) can now be simplified to the form

$$\varrho\omega u_r^* = -V_s^2 \delta\varrho_{,r}^* - H_\varphi^2 (4\pi\omega)^{-1}\{-u_{r,rr}^* + r^{-1}u_{r,r}^* - u_r^* \cdot r^{-2}\} + \\ + 2\varrho u_r^* \Omega[-(\Omega r)_{,r} - \Omega]\omega^{-1} + \varrho\delta\psi_{,r}^* , \tag{17}$$

$$\omega\delta\varrho^* + \varrho u_{r,r}^* + r^{-1}\varrho u_r^* = 0 . \tag{18}$$

We obtain from (18) and (13) (Cf. [1])

$$u_r^* = -\omega\varrho^{-1}r^{-1}\int_{r_1}^{r} x\delta\varrho^*(x)dx , \tag{19}$$

$$\delta\psi_{,r}^* = -4\pi G r^{-1}\int_{r_1}^{r} x\delta\varrho^*(x)dx . \tag{20}$$

Let us put

$$P(r) = \int_{r_1}^{r} x\delta\varrho^*(x)dx . \tag{21}$$

If we substitute (19) and (20) into (17), we obtain an equation for P

$$P_{,rr}(V_a^2 + V_s^2) - P_{,r}r^{-1}(3V_a^2 + V_s^2) + P\{-\omega^2 + 2\Omega[-(\Omega r)_{,r} - \Omega] + \\ + 4r^{-2}V_a^2 + 4\pi G\varrho\} = 0 , \tag{22}$$

where

$$V_a^2 = H_\varphi^2(4\pi\varrho)^{-1} . \tag{23}$$

If the magnetic field vanishes, $V_a = 0$, the above equation takes a form identical with Eq. (12) in [1].

If we investigate locally the stability of the system considered, in the neigbourhood of $r = r_0$, we can assume, that $P(r)$ has the periodical form

$$P(r) = \text{const} \cdot \exp ilr . \tag{24}$$

Eq. (22) gives

$$(25)\quad -l^2(V_a^2+V_s^2)-ilr_0^{-1}(3V_a^2+V_s^2)+\{-\omega^2+2\Omega[-(\Omega r)_{,r}-\Omega]+4r_0^{-2}V_a^2+4\pi G\varrho\}=0.$$

Its solution is

$$(26)\quad l=-i\frac{3V_a^2+V_s^2}{2r_0(V_s^2+V_a^2)}\mp\frac{\sqrt{\Delta}}{2(V_s^2+V_a^2)},$$

where Δ is the discriminant of Eq. (25). Considering the wave solution we can ignore the imaginary term in the expression for l. Therefore the Eq. (26) takes the form

$$(27)\quad -4l^2(V_a^2+V_s^2)^2=V_s^4r_0^{-2}+6V_a^2V_s^2r_0^{-2}+9V_a^4r_0^{-2}+-4(V_s^2+V_a^2)\{-\omega^2+2\Omega[-(\Omega r)_{,r}-\Omega]+4\pi G\varrho+4V_a^2r_0^{-2}\}.$$

The above equation is an equation for ω; from it the instability condition can be deduced. This condition is

$$(28)\quad \operatorname{sign}\omega^2=+1$$

i. e.

$$(29)\quad \operatorname{sign}\{-4l^2(V_a^2+V_s^2)^2-r_0^{-2}(V_s^4+9V_a^4+6V_a^2V_s^2)+4(V_a^2+V_s^2)[2\Omega(-(\Omega r)_{,r}-\Omega)+4\pi G\varrho+4V_a^2r_0^{-2}]\}=+1.$$

If we consider regions in which

$$(30)\quad r_0\gg l^{-1}=\frac{\lambda}{2\pi},$$

condition (29) takes the form *)

$$(31)\quad \operatorname{sign}\{-l^2(V_a^2+V_s^2)+2\Omega[-(\Omega r)_{,r}-\Omega]+4\pi G\varrho\}=+1.$$

Introducing the wavelength λ instead of wave number l, we obtain the condition for instability in the form

$$(32)\quad \lambda>\lambda_*=\pi\sqrt{\frac{V_s^2+V_a^2}{\pi G\varrho+\frac{1}{2}\Omega F}},$$

where

$$(33)\quad F=-(\Omega r)_{,r}-\Omega.$$

The instability condition given above, when applied to the uniform protogalaxy, gives the upper limit of the magnetic field intensity. If the magnetic field intensity is greater than its critical value resulting from condition (32), the axisymmetrical spiral arms with dimensions

*) This form of the instability condition is almost independent from the assumed character of the changes of the tangential magnetic field H_φ with distance (16). This is caused by the rejection of terms with r_0^{-2} in (29).

of the order of $\frac{1}{2}\lambda_*$ cannot be formed. These critical magnetic field intensities were computed for different densities of the protogalaxy, with the assumed value of sound velocity equal to 2.3 km·sec^{-1}, and the distance from the axis of symmetry equal to the distance of the Sun from the galactic centre; for this distance the value of the factor $-\frac{1}{2}F\Omega$ is equal to 0.37×10^{-30} sec^{-2}. These data are summarized in Table I.

TABLE I

ϱ \ λ_*	0.5	1.0	2.0	3.0	4.0
2.06	0	0.5	2.1	3.3	4.5
2.40	0	1.5	3.6	5.6	7.6
2.74	0.4	2.2	5.0	7.6	10.2
3.08	0.9	2.9	6.2	9.4	12.6
3.42	1.3	3.5	7.4	11.2	15.0
3.76	1.6	4.1	8.6	12.9	17.3
4.10	1.9	4.7	9.7	14.6	19.5
4.44	2.2	5.2	10.8	16.3	21.8
4.78	2.5	5.8	11.9	18.0	24.0

The upper limit of the magnetic field intensity (10^{-6} gauss) giving stability in respect to an axisymmetrical perturbation with wavelength λ_*(kpc) for different densities $\varrho(10^{-24}\,\mathrm{g\cdot cm^{-3}})$ of the protogalaxy in the neighbourhood of the Sun.

If we assume the order of magnitude of the dimensions of spiral arms as $\frac{1}{2}\lambda_* = 1$ kpc, it follows that the upper limit of the magnetic field intensity for the density of the protogalaxy in the neighbourhood of the Sun, of the order of $4.2 \times 10^{-24}\,\mathrm{g\cdot cm^{-3}}$ (actual interstellar medium density + actual star density) is equal to 1.0×10^{-5} gauss.

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A Tentative Explanation of Small Deviations Observed in Beta Spectra

by

B. KUCHOWICZ

Presented by L. INFELD on June 6, 1959

Introduction

Small deviations in beta spectra from the predictions of the present theory have been observed recently by Langer, Johnson et al. [1], [2]). The Fermi-Kurie plots for three electron spectra (^{32}P, ^{90}Y and ^{114}In) and one positron spectrum (^{22}Na) have shown deviations from linearity. An explanation of these deviations in terms of Fierz interferences would imply the opposite direction of these deviations for electron and positron spectra. Apart from other experimental evidence that the Fierz terms are vanishing, there arose, however, another objection against such an explanation. Namely both positron and electron spectra exhibited deviations corresponding to an excess of low-energy β-particles. The deviations in electron spectra have been reported before [3], [4], but no serious difficulties arose in their theoretical explanation, owing to the fact that the direction of the deviations in positron spectra was still unknown.

The investigated isotopes all decayed by pure G—T radiations though they differed in the type of forbiddenness. It may be reported with sufficient certainty that the $^{90}Y \to {}^{90}Zr$ transition is unique first forbidden ($\Delta I = 2$) while the other three are allowed ($\Delta I = 1$). The theoretical interpretation of these spectra is simple in principle because only two nuclear matrix elements appear in each case. They differ only by the appearance of the β-matrix which may be practically put equal to one. Thus only the Gamow-Teller matrix element is left for the allowed, and the unique element

$$B_{ij} = \int [\sigma_i r_j + \sigma_j r_i - \tfrac{2}{3}\delta_{ij}(\vec{\sigma}\cdot\vec{r})] \tag{1.1}$$

for the first forbidden unique transitions. The theoretical shapes of these spectra have in standard notation the well known form (with

the plus sign for electron emission and the minus sign — for positron emission)

$$F(\pm Z, E)\, pEq^2\left(1\pm\frac{b_F}{E}\right)[|C_T|^2(1+|\alpha_T|^2) \tag{1.2}$$
$$+|C_A|^2(1+\alpha_A|^2)]\times\begin{cases}|M_{GT}|^2\\ \sum|B_{ij}|^2\,\frac{1}{12}(p^2+q^2)\,,\end{cases}$$

where the upper factor is for allowed G—T transitions, and the lower — for first forbidden unique transitions.

The experimental data suggest that the Fierz term b_F is absent. However, there exists another multiplicative correction term of the same form $\left(1+\frac{b}{E}\right)$ both for electron and positron spectra. The values of the parameter b which yield a linear Fermi-Kurie plot lie in the range from 0.2 to 0.4 (in our unit system in which $\hbar = c = m_e = 1$) for the four investigated radiations. The authors of [1] and [2] do not present any theoretical explanation for the deviation observed.

Corrections of a higher order in (pr)

It seemed at first sight that by taking an exact expression for the electron wave functions one might improve the results obtained for beta spectra and obtain the desired correction term of the form $\left(1+\frac{b}{E}\right)$. The same trace technique as in the author's previous papers was used [5]-[6], but the operator A given in reference [5] was replaced by an expression completed by terms of a higher order in (pr). The exact form of this operator without any omissions was recently given in [7].

The following substitutions have now been made with regard to the functions $F_{k,i}$ defined in the Appendix II of papers [5]:

$$F_{k,i}\to F_{k,i}+F'_{k,i}+F''_{k,i}\,. \tag{2.1}$$

The primed terms are those which have been hitherto neglected and the double primed terms are the contribution from that part of the operator A which is usually considered in forbidden transitions of the order $(i+2)$. The additional terms $F'_{k,0}$ for allowed transitions have the following form:

$$\left.\begin{matrix}F'_{1,0}\\ F'_{2,0}\end{matrix}\right\} = \frac{\sqrt{E}}{2p}\left[-\frac{g_2 e^{-i\Delta_2}}{\sqrt{E+1}}\pm\frac{f_{-2}e^{-i\Delta_{-2}}}{\sqrt{E-1}}\right],$$
$$\left.\begin{matrix}F''_{1,0}\\ F''_{2,0}\end{matrix}\right\} = -\frac{3\sqrt{E}}{4p}\left[\frac{g_{-3}e^{-i\Delta_{-3}}}{\sqrt{E+1}}\pm\frac{f_3 e^{-i\Delta_3}}{\sqrt{E-1}}\right]. \tag{2.2}$$

The additional terms for forbidden transitions may be obtained in the same manner.

It has been verified by substitution that the bilinear forms of the $F_{k,i}$ functions given in [5] are subject to such insignificant changes that

the correction may be disregarded in practice. There appear certain new terms but they are multiplied by r^2 or r^4 and since in our unit system the nuclear radius r is $\sim\frac{1}{40}$, they are of no importance. This may be illustrated by a few examples. The β-decay functions L_0 and P_0 defined in [8] are now replaced, respectively, by

$$L_0+r^2(-2K^-_{c1,-2}+3K^+_{c1,3}), \tag{2.3}$$

$$P_0+r^2(2K^+_{c1,-2}-3K^-_{c1,3}), \tag{2.4}$$

where the K-functions have been generally defined in [5]. When we take the asymptotic form of the β-decay functions for $\frac{\alpha Z}{P}\ll 1$ and $pr\ll 1$, we have for both expressions given above:

$$1+r^2\left(\frac{1}{15}p^2-\frac{1}{3}\frac{p^2}{E}\xi\right) \tag{2.3a}$$

and

$$\left(1+\frac{1}{3}r^2p^2\right)\frac{p}{E}, \tag{2.4a}$$

respectively.

The corrections to beta spectra are likewise unimportant for other β-decay functions. A new way to explain the observed deviations must be sought.

The gradient corrections

It seemed encouraging to make the corrections of first order in the gradient of the lepton fields responsible for the observed deviations in beta spectra *). This kind of corrections was considered first by Gell-Mann [9] in his treatment of the "weak magnetic" effects. However, in the simplified case, discussed by Gell-Mann, these terms were of no importance because only those corrections for the high energy part of the β-spectrum were taken into account, where the "weak magnetic" effect is evident.

Let us consider the problem of unique beta transitions (pure Gamow-Teller interaction) from this point, in order to obtain theoretical corrections to the usual beta spectra. In further considerations "weak magnetic" terms will be neglected because they are of no interest for our present aim: Gell-Mann has demonstrated namely that the correction terms in beta spectra resulting from this effect change sign as we go from β^- to β^+ decay. Let us concentrate on the contribution which comes from the expansion of the interaction Hamiltonian to the first order in the gradient of the lepton fields. For simplicity, universal Fermi interaction is assumed with two phenomenological coupling constants C_v and C_A. The same notation as in the authors preceding papers [5], [6] is used. In allowed

*) Integration over the nuclear volume has been performed.

Gamow-Teller transitions, besides the usual axial vector term, two correction terms of first order are taken into account:

$$i\vec{B}\cdot\vec{V}\left(\psi_e^{+}\gamma_5(1+\gamma_5)\varphi_\nu\right) \tag{3.1}$$

and

$$i\vec{C}\,\frac{\partial}{\partial t}\left(\psi_e^{+}\vec{\sigma}[1+\gamma_5]\varphi_\nu\right). \tag{3.2}$$

The phenomenological vector constants $\vec{B}$ and $\vec{C}$ contain certain unknown nuclear matrix elements to which the axial vector as well as the vector interaction may contribute. These constants are real, if time reversal invariance for nuclear forces is assumed *). In further calculations the time derivative $\frac{\partial}{\partial t}$ may be replaced by iE_0 and the gradient — by $\frac{1}{i}(\vec{p}+\vec{q})$. Here E_0 denotes the maximum beta decay energy and $(\vec{p}+\vec{q})$ — the total lepton momentum.

Only the interference terms of first order in the vector constants $\vec{B}$ and $\vec{C}$ (which are assumed to be small quantities) are retained. The form of the beta spectrum for allowed Gamow-Teller transitions is under the assumption of the universal Fermi interaction:

$$N(E)dE = F(\pm Z, E)pEq^2\left\{|C_A|^2(1+|a_A|^2)\left|\int\vec{\sigma}\right|^2\right.$$

$$\left.-2\,\mathrm{Re}\left[C_A(1+a_A)\int\vec{\sigma}\cdot\vec{C}^*\right]E_0-\frac{2}{3}\mathrm{Re}\left[C_A(1+a_A)\int\vec{\sigma}\cdot\vec{B}^*\right]\left(E_0-\frac{1}{E}\right)\right\}dE\,. \tag{3.3}$$

It is a peculiarity of the spectrum obtained that the energy dependence may be written in the form:

$$F(\pm Z, E)pEq^2|C_A|^2(1+|a_A|^2)\left|\int\vec{\sigma}\right|^2\left\{1-(b+c)E_0+\frac{b}{E}\right\}dE \tag{3.4}$$

with the abbreviations:

$$b=\frac{\frac{2}{3}\,\mathrm{Re}\left[C_A(1+a_A)\int\vec{\sigma}\cdot\vec{B}^*\right]}{|C_A|^2(1+|a|^2)\left|\int\vec{\sigma}\right|^2}\,,$$

$$c=\frac{2\,\mathrm{Re}\left[C_A(1+a_A)\int\vec{\sigma}\cdot\vec{C}^*\right]}{|C_A|^2(1+|a_A|^2)\left|\int\vec{\sigma}\right|^2}\,. \tag{3.5}$$

It is now evident that the empirical correction introduced by Langer, Johnson et al. [1], [2] may be explained in the framework of the usual beta theory after introducing the gradient corrections. It can be easily estimated that the experimental spectrum shape $\sim\left(1+\frac{0.3}{E}\right)$ can be obtained even for small values of the parameters b and c [e. g. for $b=0.08$ and $c=0.067$ if $E_0=5$], i. e. for a small admixture of correction terms.

*) Two further terms of first order are *a priori* possible but have been discarded.

The same calculations have been performed in the case of first forbidden unique beta transitions, where instead of the vector constants $\vec{B}$ and $\vec{C}$ the tensor constants T_{ij} and C_{ij} are taken into account:

$$
\begin{aligned}
T_{ij} &= B_i r_j + B_j r_i - \tfrac{2}{3}\delta_{ij}\vec{B}\cdot\vec{r}\,, \\
C_{ij} &= C_i r_j + C_j r_i - \tfrac{2}{3}\delta_{ij}\vec{C}\cdot\vec{r}\,.
\end{aligned}
\tag{3.6}
$$

The correction term of tensor type $iD_{ij}\nabla_i[\psi_e^+\sigma_j(1+\gamma_5)\varphi_\nu]$ is added also to the Hamiltonian. In the subsequent formulae the symmetrized tensor D_{ij}^s is used:

$$
D_{ij}^s = D_{ij} + D_{ji} - \tfrac{2}{3}\delta_{ij}(D_{11}+D_{22}+D_{33})\,. \tag{3.7}
$$

In the formulae below, B_{ij} is the unique matrix element in standard notation. The spectrum obtained exhibits corrections of a type similar to that of corrections in the allowed spectrum:

$$
\begin{aligned}
N(E)dE = F(\pm Z, E)pEq^2 \Big\{ & |C_A|^2(1+|\alpha_A|^2)\sum |B_{ij}|^2 \cdot \frac{1}{12}(p^2+q^2) \\
& + 2\,\mathrm{Re}\Big\{C_A(1+\alpha_A)\sum [B_{ij}T_{ij}^*]\Big[-\frac{1}{30}\Big(q^3+\frac{p^4}{E}\Big)+i\frac{q^2\alpha Z}{12}\Big(\frac{p^4}{4E}+\frac{1}{3E}+\frac{p^2}{4}\Big)\Big]\Big\} \\
& - 2\,\mathrm{Re}\Big[C_A(1+\alpha_A)\sum [B_{ij}C_{ij}^*]\Big]\cdot\frac{1}{12}E_0(p^2+q^2) \\
& + 2\,\mathrm{Re}\Big\{C_A(1+\alpha_A)\sum [B_{ij}D_{ij}^{s*}]\Big[\frac{\alpha Z p^3}{40}\Big(1+\frac{7p^2}{4E}+\frac{7p}{3E}\Big)-i\frac{7}{120}(q^2-p^2)\Big]\Big\}\Big\}dE\,.
\end{aligned}
\tag{3.8}
$$

Theoretical interpretation of this formula is rendered more difficult because of the great number of matrix elements, even though only the interference terms are retained. Simplifying assumptions may reduce the formula to the first three terms only and it can now be written in the form:

$$
\begin{aligned}
N(E) = F(\pm Z, E)pEq^2|C_A|^2(1+|\alpha_A|^2)\sum |B_{ij}|^2\cdot\frac{1}{12}(p^2+q^2) \\
\times\left[1-\frac{1}{5}b'\frac{q^3+\frac{p^4}{E}}{p^2+q^2}-c'E_0\right]dE
\end{aligned}
\tag{3.9}
$$

with the abbreviations:

$$
\begin{aligned}
b' &= \frac{3\,\mathrm{Re}\,[C_A(1+\alpha_A)\sum B_{ij}T_{ij}^*]}{10\,|C_A|^2(1+|\alpha_A|^2)\sum |B_{ij}|^2}\,, \\
c' &= \frac{3\,\mathrm{Re}\,[C_A(1+\alpha_A)\sum B_{ij}C_{ij}^*]}{10\,|C_A|^2(1+|\alpha_A|^2)\sum |B_{ij}|^2}\,.
\end{aligned}
\tag{3.10}
$$

The spectrum shape resembles now the form of the allowed spectrum.

The numerical calculations of the coefficients b, c, b' and c' will be possible on the basis of nuclear model assumption for some simple decays, e.g. for mirror nuclei. It seems therefore desirable to investigate experimentally those decays in order to be able to obtain a quantitative comparison of theory with experiment. Slight changes in $e-\nu$ angular correlation may also be investigated, though experimental demonstration will be difficult.

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Abstract

The small deviations in beta spectra observed experimentally [1]-[4] could have been linearized only by introducing an empirical Fierz-like correction term with the same sign for electron and positron spectra. In the present paper terms of this type are obtained under the simple assumption of a semiphenomenological nature. As the $A-T$ interference in Gamow-Teller transitions is now rather considered to be impossible, there remains only the supposition that the experimental phenomena might be explained by the suppression of the usually dominant matrix elements. The scheme considered is based on Gell-Mann's ideas. It is possible to calculate some new terms in the $e-\nu$ angular correlation, polarization formulae etc., however, the intricate form of the expressions involved seems hardly to render possible any definite conclusions.

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On the Mass Spectrum of Baryons

by

L. ŁUKASZUK

Presented by W. RUBINOWICZ on June 12, 1959

The mass spectrum of baryons was examined within the scheme of strong interactions given by W. Królikowski [1]. The $\Sigma-\Lambda$ mass difference is only caused by interaction between baryons and the π-meson field. The Hamiltonian of interaction with the π-mesons violates the "global symmetry" proposed by Gell-Mann [2] and Schwinger [3].

The $\Sigma-\Lambda$-mass difference is equal to $160m_e$, $\Xi-N=750m_e$, $\Lambda-N=317m_e$ in good agreement with experimental data. Besides, the assumption is made that the "bare" mass spectrum has a form dependent on the "strangeness" S: $m(S)=m+S\mu$, where $\mu=212m_e$ is the mass of μ-meson. The calculus is performed in non relativistic approximation, $\frac{f_{0\pi}^2}{4\pi}=0.08(G_{0\pi}^2=15)$, for K-interaction two extreme values are put: $G_{0K}^2=2$ and 8. The cut-off for π-interaction is $k_{\max}=3.5\mu_\pi$. In K-interaction we have: $k_{\max}=4.9\mu_\pi$ and $3.5\mu_\pi$ (!) for $G_{0K}^2=2$ and 8. The symmetrical form of interaction, involving only one coupling constant from experiment, is proposed. In this case the common cut-off $k_{\max}=3.5\mu_\pi=1\mu_K$ for π and K-interaction accounts for the mass spectrum of baryons. (The cut-off obtained by Chew and Low is $k_{\max}=5.6\mu_\pi$, so ours seems to be too small).

In the scheme of strong interactions given by W. Królikowski [1] the following Hamiltonian was proposed:

$$\mathcal{H}=\mathcal{H}^{B\pi}+\mathcal{H}^{BK},$$

where

$$\mathcal{H}^{B\pi}=ig_0^\pi\bar{B}\gamma_5(\vec{\tau}+\xi_\alpha^*\vec{\tau}_{\alpha\beta}\xi_\beta)\cdot B\cdot\vec{\pi}$$

and

$$\mathcal{H}^{BK}=ig_0^K\bar{B}\gamma_5\xi_\alpha B(CK)_\alpha+h\cdot c., \text{ where } C=-\tau_2.$$

We shall modify the Hamiltonian $\mathcal{H}^{BK}$. In the first place, let us remark that

$$Q^B = e(T_3^B + \tfrac{1}{2}N^B - \tfrac{1}{2}S^B),$$

where

$$\vec{T}^B = \tfrac{1}{2}\int d_3x B^*(\vec{\tau} + \xi_\alpha^*\vec{\tau}_{\alpha\beta}\xi_\beta)B\,, \qquad N^B = \int d_3x B^*B$$

and

$$S^B = \int d_3x B^*\xi_\beta^*\xi_\beta B\,.$$

We thus see, that $\vec{T}^B$ is responsible for the π-interaction. The assumption that the K-interaction is connected with the rest of Q^B, namely with $N^B - S^B$ seems most natural. So, instead of ξ_a in $\mathcal{H}^{BK}$ we put $\xi_a - \xi_\beta^*\xi_a\xi_\beta = \xi_a S$, where S—the "strangeness" operator before the second quantisation (ξ_a plays in the K-interaction a similar role as $\vec{\tau}$ in the π-interaction).

Finally, we propose the Hamiltonians:

$$\begin{aligned}\mathcal{H}^{B\pi} &= ig_0^\pi \bar{B}\gamma_5(\vec{\tau} + \xi_\alpha^*\vec{\tau}_{\alpha\beta}\xi_\beta)B\cdot\vec{\pi}\\ \mathcal{H}^{BK} &= ig_0^K \bar{B}\gamma_5(\xi_a - \xi_\beta^*\xi_a\xi_\beta)B(CK)_a + h.\,c. = ig_0^K \bar{B}\gamma_5\xi_a\xi_\beta^*\xi_\beta B(CK)_a + h.\,c.\end{aligned} \tag{1}$$

Putting in (1) the components of the baryon field, we have (see [1]):

$$\mathcal{H}^{B\pi} = ig_0^\pi(\bar{N}\gamma_5\vec{\tau}N + 2i\bar{\vec{\Sigma}}\gamma_5\times\vec{\Sigma} + \bar{\Xi}\gamma_5\vec{\tau}\Xi)\cdot\vec{\pi} \tag{1a}$$

and

$$\mathcal{H}^{BK} = \frac{ig_0^K}{\sqrt{2}}(\bar{N}\gamma_5\Lambda^0 + \bar{N}\gamma_5\vec{\tau}\cdot\vec{\Sigma} + 2\bar{\Lambda}^0\gamma_5\Xi C - 2\bar{\vec{\Sigma}}\gamma_5\Xi C\cdot\vec{\tau})K + \text{h.c.}\ ^*) \tag{1b}$$

We shall now show what mass spectrum of baryons is obtained assuming they have a common "bare" mass m. The mass correction δm_N for the second order approximation, is, symbolically:

$$\delta m_N = \Big[\underset{n}{\rightarrow}\overset{\pi^0}{\bigcirc}\underset{n}{\rightarrow} + 2\,\underset{n}{\rightarrow}\overset{\pi^-}{\bigcirc}\underset{p}{\rightarrow}\Big] + \Big[\underset{n}{\rightarrow}\overset{K^0}{\bigcirc}\underset{\Sigma^0}{\rightarrow} + \tfrac{1}{2}\,\underset{n}{\rightarrow}\overset{K^+}{\bigcirc}\underset{\Sigma^-}{\rightarrow} + \tfrac{1}{2}\,\underset{n}{\rightarrow}\overset{K^0}{\bigcirc}\underset{\Lambda^0}{\rightarrow}\Big]$$

or $\delta m_N = 3\delta\pi + 2\delta K$ because of equal "bare" baryon masses.

Similarly we get:

$$\begin{aligned}\delta m_\Lambda &= 5\delta K\\ \delta m_\Sigma &= 8\delta\pi + \delta m_\Lambda\\ \delta m_\Xi &= 3\delta\pi + 8\delta K\\ \delta m_N &= 3\delta\pi + 2\delta K\,.\end{aligned} \tag{2}$$

*) We see, that $\frac{1}{2}\frac{g_{0K}^2}{4\pi} = G_{0K}^2$.

To account for $\Sigma - \Lambda$ mass difference one must put $\delta\pi \simeq 20 m_e$, but at the same time $\delta K \simeq 100 m_e$ (because $m_\Xi - m_N = 750 m_e$).

It has been found that the K-interaction with $\frac{g_{0K}^2}{4\pi} \simeq 4$ is too weak to give such a change of mass.

The suggestion, due to W. Królikowski, has been made that the mass differences among baryons with different S should have partly the same origin as the mass of μ-meson ($\mu = 212 m_e$).

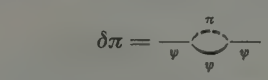
In this case "bare" masses should have approximately a spectrum: $m(S) = m + S\mu$, where S – the "strangeness", m – the "bare" mass of nucleon ($S = 0$), and $\mu = 212 m_e$.

Taking advantage of this proposal, we have:

$$\text{(2a)} \qquad \begin{aligned} \delta m_N &= 3\delta\pi + 2\delta K\,, \\ \delta m_\Lambda &= 5\delta K + \mu\,, \\ \delta m_\Sigma &= 8\delta\pi + \delta m_\Lambda\,, \\ \delta m_\Xi &= 3\delta\pi + 8\delta K + 2\mu\,. \end{aligned}$$

We made an approximation putting in (2a) δK and $\delta\pi$ from (2), although we now have no common "bare" mass. Thus the diagram for δK may be obtained from that for $\delta\pi$, if we replace μ_π, g_0^π by μ_K, g_0^K respectively. This is not the only simplification in the calculus. We make calculations in non-relativistic approximation putting $f_{0\pi}^2$, f_{0K}^2 equal to the renormalized ones obtained from the processes with nucleons. In other words we assume that the renormalization of $g_0^{\pi,K}$ does not lead to different coupling constants for different baryon components.

Now, any diagram giving $\delta\pi$ has the form:



corresponding to an interaction: $\mathcal{H}_I = i g_0^\pi \bar{\psi}\gamma_5\psi\cdot\pi$, where ψ is a fermion field with mass m, and π is a real, pseudoscalar field with mass $\mu_\pi = 273 m_e$. Next, in non-relativistic calculus we use the Hamiltonian:

$$H_{\text{n.r.}} = H_0 + H_I = \frac{p^2}{2m} + \int \omega(k) N(k)\, d_3k + \frac{f_{0\pi}}{\mu_\pi}\sigma_i \int d_3x\, \varrho^\pi(x - x')\, \partial_i' \pi(x')\,,$$

where, as usual:

$$H_0 = \frac{p^2}{2m} + \int \omega(k) N(k)\, d_3k\,, \qquad \text{and} \qquad \omega = \sqrt{k^2 + \mu_\pi^2}$$

(we put here $\hbar = c = 1$). Passing to the n.r. H_I we omitted the part with $\pi^2(x)$, involving two virtual mesons in mass-correction). Using $H_{\text{n.r.}}$ we get in the perturbation theory:

$$\text{(3)} \qquad \Delta E = \langle \psi_0 | H_I \frac{1}{E_0 - H_0} H_I | \psi_0 \rangle\,,$$

where ψ_0 fulfills the conditions:

$$H_0|\psi_0\rangle = E_0|\psi_0\rangle\,, \qquad N|\psi_0\rangle = 0\,.$$

Hence $\psi_0 = \frac{1}{(2\pi)^{3/2}} e^{i\vec{p}\vec{x}}\Phi_0$; of course, $N(k)\,|\Phi_0\rangle = 0$.

From (3) we have after substitution:

$$\Delta E = \langle\Phi_0| \frac{f_{0\pi}^2}{\mu_\pi^2}\int d_3x\, e^{-i\vec{p}\vec{x}} d_3x'\, d_3x''\, \varrho^\pi(x-x'')\,\sigma_r\sigma_i\,\partial_r''\pi(x'') \times \frac{1}{E_0-H_0}\,\varrho^\pi(x-x')\,\partial_i'\pi(x')\,e^{i\vec{p}\vec{x}}|\Phi_0\rangle,$$

where

$$\varrho^\pi(x-x') = \frac{1}{(2\pi)^3}\int d_3k\, v(k)\, e^{i\vec{k}(\vec{x}-\vec{x}')}, \qquad \text{with} \qquad v(0)=1\,,\ v(\vec{k}) = v(k)$$

and

$$\pi(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d_3k}{\sqrt{2\omega}}\, a(k)\, e^{i\vec{k}\vec{x}} + \text{h.c.}$$

After simple calculations we get an expression for ΔE:

$$\Delta E = \frac{f_{0\pi}^2}{\mu_\pi^2}\frac{1}{(2\pi)^3}\int \frac{(\vec{\sigma}\vec{k})(\vec{\sigma}\vec{k})\,v^2(k)\,d_3k}{\left[\dfrac{\vec{p}^2}{2m} - \dfrac{(\vec{p}-\vec{k})^2}{2m} - \omega\right]2\omega}\,.$$

If we define

$$a = -\frac{f_{0\pi}^2}{\mu_\pi^2}\frac{1}{2}\frac{m}{(2\pi)^2} \tag{4}$$

and introduce polar co-ordinates, we get:

$$\Delta E = 2a\int_{-1}^{+1}\int_0^\infty \frac{k^4v^2dk\,dz}{[k^2+2m\omega-2pkz]\,\omega}\,.$$

But $|2pkz| \ll k^2+2m\omega$, because for $k>2p$, $k^2 \gg 2kpz$ and for $k\simeq p$, $2pkz \simeq 2p^2z \ll 2m\mu_\pi$ (n.r. nucleon). Taking this into account, we get:

$$\Delta E = 2a\int \frac{k^4v^2dk\,dz}{[k^2+2m\omega]\left[1-\dfrac{2pkz}{k^2+2m\omega}\right]\omega}$$

$$= 2a\int \frac{k^4v^2dk\,dz}{\omega(k^2+2m\omega)}\left[1+\frac{2pkz}{k^2+2m\omega}+\frac{(2kpz)^2}{(k^2+2m\omega)^2}+\ldots\right].$$

Hence

$$\Delta_p E = \frac{2\cdot 8}{3}\,ap^2\int \frac{k^6v^2dk}{\omega(k^2+2m\omega)^3}$$

and

$$\delta m = -\frac{2m^2}{p^2}\,\Delta_p E = -\frac{32}{3}\,am^2\int \frac{v^2k^6dk}{\omega(k^2+2m\omega)^3}\,.$$

We now put

$$v(k) = \begin{cases} 1 & \text{for} \quad k \leqslant a\mu_\pi < m \\ 0 & \text{for} \quad k > a\mu_\pi \end{cases}$$

and

$$\delta m = -\frac{32am^2}{3}\int_0^{a\mu_\pi} \frac{k^6 dk}{\omega(k^2+2m\omega)^3} = -\frac{32}{3}am^2 \int_0^a \frac{x^6 dx}{\sqrt{x^2+1}\left(x^2+2\frac{m}{\mu_\pi}\sqrt{x^2+1}\right)^3}.$$

The integral

$$I = \int \frac{x^6\,dx}{\sqrt{x^2+1}\,(x^2+2\delta\sqrt{x^2+1})^3}, \qquad \text{with} \qquad \delta = \frac{m}{\mu_\pi},$$

is an elementary one, and may be transformed into a rational form of integrand (5):

$$(5) \qquad I = \int_0^a \frac{x^6 dx}{\sqrt{x^2+1}\,(\sqrt{x^2+1}+\delta+\sqrt{\delta^2+1})^3\,(\sqrt{x^2+1}+\delta-\sqrt{\delta^2+1})^3}$$

and, after substitution $\sqrt{x^2+1} = x+t$, we get

$$I = \int \frac{dt(1-t^2)^6}{t\left[1+t^2+2t\left(\delta+\sqrt{\delta^2+1}\right)\right]^3\left[1+t^2+2t\left(\delta-\sqrt{\delta^2+1}\right)\right]^3}.$$

The value of $\delta = \dfrac{m}{\mu_\pi}$ depends on the "bare" mass m. Now, if (2a) led to a good mass spectrum, we should have $\delta\pi \simeq 20m_e$ $\delta K \simeq 56m_e$, so, $m = m_N - \delta m_N = 1670m_e$.

Finally $\dfrac{m}{\mu_\pi} = 6.1$, and $\dfrac{m}{\mu_K} = 1{\cdot}7$.

Putting in (5) $k_{\max} = a\mu_\pi \simeq 3.5\mu_\pi$ we get, after numerical calculations:

$$\delta\pi \simeq 20m_e \left(\frac{f_{0\pi}^2}{4\pi} \simeq 0.08\right).$$

Similarly, cutting at $k_{\max} \simeq 1.4\mu_K$, and $1\mu_K$ we get $\delta K \simeq 55m_e \left(\dfrac{g_{0K}^2}{4\pi} \simeq 4 \text{ and } 15\right)$. With such values for $\delta\pi$, δK, we get a mass-spectrum being in good agreement with experimental data as shown below:

TABLE

Particles	Theoretical value Δm in m_e	Experimental value Δm in m_e
$\Xi - N$	754	750 (for $\Xi^- - N$)
$\Sigma - \Lambda$	160	159 (for $\Sigma^- - \Lambda^0$)
$\Lambda - N$	317	342 (for $\Lambda^0 - N$)

Let us now write the Hamiltonians (1):

$$\mathcal{H}^{\text{int}} = \mathcal{H}^{BK} + \mathcal{H}^{B\pi} = i\bar{B}\gamma_5[g_{0\pi}(\vec{\tau} + \xi_a^*\vec{\tau}_{a\beta}\xi_\beta)\vec{\pi} + g_{0K}\xi_a\xi_\beta^*\xi_\beta(CK)]_a B + \text{h.c.}$$

Taking into account, that $\vec{T}^B = \vec{\tau} + \xi_a^*\vec{\tau}_{a\beta}\xi_\beta$ and $S^B = \xi_\beta^*\xi_\beta$ (T^B, S^B — the total isospin and strangeness of baryon field before the second quantisation), we have:

$$\mathcal{H}^{\text{int}} = i\bar{B}\gamma_5[g_{0\pi}\vec{T}^B\vec{\pi} + g_{0K}(CK)_a\xi_a S^B]B + \text{h.c.}$$

Now we put $g_{0K} = g_{0\pi} = g_0$.

So the full Hamiltonian is:

$$(6) \quad \mathcal{H} = \mathcal{H}_0 + \mathcal{H}^{\text{int}} + \mu\bar{B}S^B B$$
$$= \mathcal{H}_0 + \bar{B}\{i\gamma_5 g_0[\vec{T}^B\vec{\pi} + (CK)_a\xi_a S^B] + \mu S^B\}B + \text{h.c.}$$

Because for $\frac{g_{0K}^2}{4\pi} = 15 = 2G_{0K}^2$ we obtained in K-interaction cut-off equal to that for π-interaction, the Hamiltonian (6) accounts for the mass spectrum of baryons.

The author is indebted to Professor W. Królikowski for suggesting the problem and for helpful advice.

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THEORETICAL PHYSICS

On the π^+ Decay of a Hyperfragment

by

A. DELOFF, J. SZYMAŃSKI and J. WRZECIONKO

Presented by W. RUBINOWICZ on June 17, 1959

Within recent years numerous examples of hyperfragment decay have been reported [1], [2]. Hyperfragments are known to decay by three modes a) nonmesic decay, b) mesic decay with the π^- or π^0 emission, c) "anomalous decay" (leptonic decay, π^+ emission). In this paper the π^+ emission from a hyperfragment is considered and the definition "anomalous" is introduced here by the authors for two reasons. Firstly, according to existing experimental evidence such a decay is found to be extremely rare. Among several hundreds identified hyperfragment decays only one π^+ decay was observed [3]. The second reason is purely theoretical. The π^+ emission can be easily understood given the concept of the virtual $\Lambda \leftrightarrow \Sigma^+$ exchange inside the hyperfragment which precedes the decay of the hyperon while in the Σ^+ state. The π^+ emission is then the decay of a virtual Σ^+-hyperfragment contrary to all ordinary Λ-hyperfragment decay as quoted under a) and b) above. This seems to be the most plausible explanation of the process. The other possibility, that of an ordinary mesic decay followed by the charge exchange reaction is very unprobable because, in the low energy region, the cross-section for the $\pi^0 + p \rightarrow \pi^+ + n$ reaction is very small. The authors consider only the first approach and try to estimate the π^+ decay frequency relative to the number of the π^- decays. Knowledge of the experimental value of this branching ratio would yield much information concerning the baryon interaction and the hyperfragment structure. Unfortunately, at the present time, the experimental data are not sufficient to estimate even the order of magnitude of this branching ratio. For this reason we do not try to draw more far-reaching conclusions from the present calculation and confine our attention only to a determination of the order of magnitude of the relative frequency of the π^+ decay. Calculations were performed under the assumption that only the π-coupling is important in the $N\Lambda\Sigma$ interaction. The K-coupling is discussed without calculation.

We came to the conclusion that when both couplings are present the branching ratio can be slightly bigger, but in no case can its order of magnitude be changed.

In general, the ratio of the emitted π^+ to the π^- in a hyperfragment decay should be a sensitive function of the nuclear structure and should vary for different hyperfragments. Let us give some examples which can serve as arguments. The emission of π^+ from a hyperfragment requires a change of a proton into a neutron inside the core. If this process gives rise to a strong disturbance of the core the π^+ emission is very unprobable. We can take ${}_\Lambda He^5$ as an example. The π^+-decay of this hyperfragment would spoil such a very stable structure as the α particle. In the opposite case, where the π^+ emission leads to a final nucleus which is a mirror partner of the core, the probability of the π^+ decay can be relatively great, because of the big value of the coefficient of fractional parentage (eg. the reaction ${}_\Lambda He^4 \to H^3 + n + \pi^+$). Because of the simplicity of the calculation the lightest hyperfragments are the most convenient for discussion. The authors have chosen ${}_\Lambda He^4$ for their particular considerations, since it is the lightest hyperfragment in which the π^+ emission does not smash the core. For this hyperfragment several simplifications in the calculation are possible and do not introduce a significant error.

Method of calculation

The bound state $He^3 + \Lambda$ is considered and the baryon strong interactions are assumed in a form given by the following Hamiltonian density (notation adopted from [4])

$$H = H_N + H_\Lambda ,$$

where

$$H_N = -\frac{f_N}{\mu} \bar{N} \vec{\sigma} \tau_a N \nabla \pi_a ,$$

$$H_\Lambda = -\frac{f_\Lambda}{\mu} \bar{\Sigma}_a \vec{\sigma} \Lambda \nabla \pi_a + \text{h. c.} ,$$

where μ is the mass of the π meson, and f_N and f_Λ are the pseudovector πN and $\pi \Lambda$ — coupling constants respectively.

We take the Hamiltonian density responsible for the weak process of the $\Lambda(\Sigma)$ decay in the form [5]

$$H_\Sigma^w = g_\Sigma \bar{n}[(1 + \beta_\Sigma \vec{\sigma} \nabla)\pi^{+*}]\Sigma^+ + g'_\Sigma \bar{p}[(1 + \beta'_\Sigma \vec{\sigma} \nabla)\pi^{0*}]\Sigma^+ ,$$

$$H_\Lambda^w = g_\Lambda \bar{p}[(1 + \beta_\Lambda \vec{\sigma} \nabla)\pi^{-*}]\Lambda + g'_\Lambda \bar{n}[(1 + \beta'_\Lambda \vec{\sigma} \nabla)\pi^{0*}]\Lambda .$$

In the lowest order of the perturbation theory we have three graphs for the π^+ emission proceeding through the virtual reaction $\Lambda \to \Sigma^+ + \pi^-$.

The application of the perturbation theory seems to be justified here, since the energy of the virtual π does not exceed 100 MeV. On the other hand, one gets, as is well known, good results by this method for the πN scattering in just this energy range. Experimental fit is available

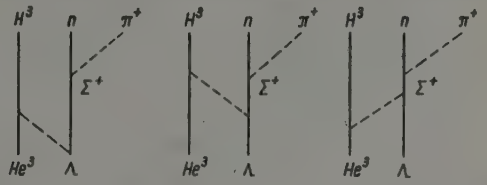

Fig. 1

for the PV theory with the coupling constant $f^2 \approx 0.05$ [6]. The general form of the matrix element for our process is

$$M = \langle f|H_\Sigma^w \frac{1}{E-H_0} H_N \frac{1}{E_0-H} H_\Lambda|i\rangle + \langle f|H_N \frac{1}{E-H_0} H_\Sigma^W \frac{1}{E-H_0} H_\Lambda|i\rangle + \langle f|H_W^\Sigma \frac{1}{E-H_0} H_\Lambda \frac{1}{E-H_0} H_N|i\rangle ,$$

where E is the total energy of the system. In the initial state $|i\rangle$ we have Λ coupled with the core He^3, and in the final state $|f\rangle$ three interacting particles H^3, n and π^+ occur. In the matrix element we replace the exact Λ wave function by an approximate one, proportional to $\delta(\vec{p}_\Lambda)$. This corresponds to the constant density inside the hyperfragment. This procedure does not lead to a significant error, since all the graphs considered are of short range in comparison with the hyperfragment radius. In the final state we neglect the interaction of decay products. It is to be expected that the error so introduced will be partly cancelled out when the branching ratio is calculated, since the π^- emission is calculated in the same approximation. The error in both channels should be very close due to the identical form of the interaction, this being a consequence of the charge independence.

After these simplifications we have for the probability of the reaction ${}_\Lambda He^4 \to H^3 + n + \pi^+$ the following expression

$$W(\pi^+) = \frac{8}{\pi(R\mu)^3} f_N^2 f_\Lambda^2 \frac{g_\Sigma^2}{4\pi} \frac{M_N}{M_\Lambda} \int_0^{v_{\max}} dv\, l\, v^6 (1+\beta_\Sigma^2 k^2) \times \left[\frac{2(\Delta M+\omega) + \frac{1}{2}v^2\left(\frac{1}{M_t}+\frac{1}{M_\Sigma}\right)}{\omega\left(\omega+\frac{1}{2}\frac{v^2}{M_t}\right)\left[\Delta M+\frac{1}{2}v^2\left(\frac{1}{M_t}+\frac{1}{M_\Sigma}\right)\right](\Delta M+\omega)} \right]^2 ,$$

where the following abbreviations are introduced:

$$\Delta M = M_\Sigma - M_\Lambda\,, \qquad R - \text{hfr. radius}\,,$$
$$\omega = (\mu^2 + k^2)^{1/2}\,,$$
$$k^2 = (1+\mu/M)^2 v^2 + l^2; \qquad M = M_N + \mu\,,$$
$$l = \left(\frac{2M_N\mu}{M_N+\mu}\right)^{1/2}\left[Q - \frac{1}{2}v^2\left(\frac{1}{M}+\frac{1}{M}\right)\right]^{1/2},$$
$$v_{\max} = \left[\frac{2Q\,M_t(M_N+\mu)}{M_t+M_N+\mu}\right],$$
$$M_t - \text{is the mass of } H^3$$

and Q is the total energy released in the decay. The integral in (1) was computed numerically, and for $W(\pi^+)$ we get

$$W(\pi^+) = \frac{8}{\pi(\mu R)^3} f_N^2 f_\Lambda^2 \frac{g_\Sigma^2}{4\pi}\frac{M_N}{M_\Lambda}(3.1+\beta_\Sigma^2 6.8)\,. \tag{2}$$

Similarly we can get the probability of the hyperfragment decay with π^--emission. In our approximation it reduces to the probability of the free Λ decay. We have

$$W(\pi^-) = \frac{g_\Lambda^2}{4\pi} 2k_\Lambda(1+\beta_\Lambda^2 k_\Lambda^2)\frac{M_N}{M_\Lambda}\,, \tag{3}$$

where k_Λ is the momentum of π^- emerging in the Λ decay in the rest system of Λ.

The considered $W(\pi^+)/W(\pi^-)$ branching ratio depends on the ratio the weak coupling constants g_Σ^2/g_Λ^2. The latter can be eliminated because it occurs in the Σ^+ and Λ lifetime ratio. As a matter of fact

$$\frac{\tau_\Lambda}{\tau_\Sigma} = \frac{g_\Sigma^2}{g_\Lambda^2}\frac{1+\beta_\Sigma^2 k_\Sigma^2}{1+g_\Lambda^2 k_\Lambda^2}\cdot\frac{k_\Sigma}{k_\Lambda}\cdot 1.28\,.$$

Expressing g_Σ^2/g_Λ^2 by τ_Λ/τ_Σ we have for the branching ratio

$$B \equiv \frac{W(\pi^+)}{W(\pi^-)} = \frac{4}{\pi}(R\mu)^{-3} f_N^2 f_\Lambda^2 \frac{\tau_\Lambda}{\tau_\Sigma}\frac{3{,}1+\beta_\Sigma^2 6{,}8}{1+\beta_\Sigma^2 k_\Sigma^2}\cdot\frac{k_\Lambda}{k_\Sigma}\cdot 0.78\,.$$

For $f_\Lambda^2 \approx f_N^2 = 0.085$ the branching ratio B is of the order 10^{-2}. The numerical values of B for several β_Σ and R are given in the table below.

Branching ratio $\times 10^{-2}$		
R in Fermi units	S-state $\beta_\Sigma = 0$	P-state $\beta_\Sigma = \infty$
1.8	1.7	1.9
2	1.2	1.4
1.6	4.0	4.5

In Fig. 2 the momentum distribution of H^3 is given in the c. m. system. In our case the curve differs considerably from the one where the π^--emission takes place. For the π^--emission the main part of the energy is removed by the meson, whereas in the π^+ emission the most probable decays are those where low energetic mesons are emitted. The results given above are obtained with only the π coupling taken into account.

If the coupling with K is introduced our result for the branching ratio will slightly change. Let us estimate this correction. In the same order of the perturbation theory 9 graphs with K give a contribution to our process. These are graphs with only one virtual K involved and the number of graphs is here greater than in Fig. 1 because graphs with the virtual K decaying into two pions should be added. However, all these graphs will give a small contribution due to the large mass of K and small value of the coupling constant. If only the K-field were responsible for the Λ-core interaction, then the considered branching ratio could be expected to be of the 10^{-3} order (the KN coupling constant $\approx 2-3$). In the case when both π and K interactions are present the contribution of K-interference terms does not exceed 2/5 of the π contribution.

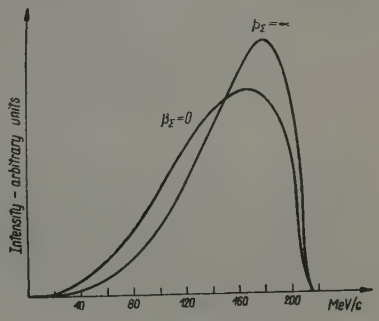

Fig. 2

It seems probable that the measurement of the branching ratio $W(\pi^+)/W(\pi^-)$ would throw some light on the question of what kind of interaction is important in the hyperfragment.

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EXPERIMENTAL PHYSICS

Potential Scattering of Neutrons in the Resonance Region

by

W. RATYŃSKI, J. TURKIEWICZ, P. ŻUPRAŃSKI

Presented by A. SOŁTAN on June 19, 1959

Introduction

The main purpose of this paper is to develop a simple and rapid method for the estimation of potential-scattering cross-sections. Seth's well-known method [1] was extended over the whole neutron spectrum of a reactor (in the region above Cd cut-off with Boron detection). Consider, after Seth, a hypothetical element with no resonance structure. In such an element potential scatering is the only way of removing neutrons from the incident beam. Thus, the logarythmic plot of transmission T against sample thickness n represents a straight line with a slope proportional to the potential scatering cross-section σ_P. However, each element has actually a resonance structure and therefore the above simple picture is not valid.

The existence of resonances makes the plot to curve upward for small thicknesses. For sufficiently thick samples, however, $\ln T$ is a linear function of their thicknesses, because the first layers of the sample are effective enough to remove from the beam all neutrons at their resonance energies. Neglecting the effect of interference between potential and resonance scattering [2] the slope of the linear part may be interpreted as a measure of the potential scattering cross-section.

Apparatus

The neutron beam was obtained in the EWA Polish reactor at Świerk near Warsaw, operating at a power of 2 MW.

The neutron beam was defined by an iron collimator and had a diameter of 5.6 mm.

A proportional counter filled with BF_3 to the pressure of 600 mm Hg was used as a neutron detector. Pulses from the counter were fed to a conventional amplifier and then counted by a fast scalar unit. The distance from the sample to the neutron detector was 70 cm.

Results

The results of transmission measurements for aluminium, silver and bismuth are shown in Fig. 1.

For silver which has a great number of low lying resonances the plot shows the characteristic deviation from a straight line for small thicknesses. In Al and Bi, on the other hand, such resonances do not occur in the

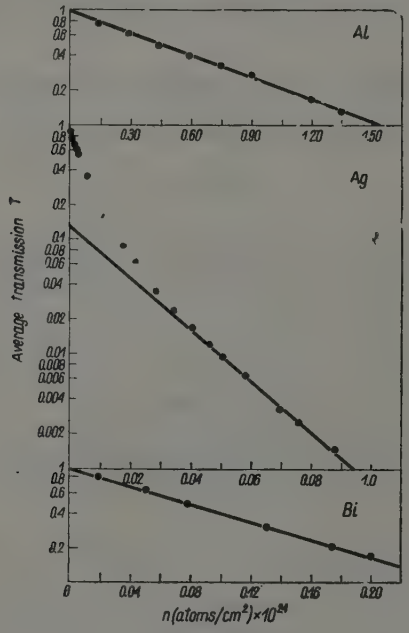

Fig. 1

energy region considered and the plots are well approximated by the straight line throughout the whole range of the thicknesses investigated.

The straight lines were drawn through the experimental points using the least square method. Calculated cross-sections are given in the Table.

TABLE

Sample	σ_P	
	our results	BNL results
Al	1.4 ± 0.1	—
Ag	5.3 ± 0.4	5.7 ± 0.4
Bi	8.9 ± 0.4	10.2 ± 0.2

The errors shown are only statistical ones.

The results obtained for bismuth differ significantly from those of Brookhaven National Laboratory [3].

It is interesting to note that our result is in better agreement with the theoretical predictions based on the cloudy crystal ball model [4], [5].

The authors wish to thank Professor A. Sołtan for many stimulating discussions and for his critical comments at all stages of the work.

Summary

Transmission measurements for various elements were carried out using a continuous neutron spectrum of a reactor. The plots of $\ln T$ vs. sample thickness were obtained for Al, Ag and Bi.

It seems reasonable to interpret the slopes of these curves for large sample thicknesses as potential-scattering cross-sections.

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Raman Spectra of Bromoform Solution in Diethylamine

by

S. BRAHMS

Presented by A. JABŁOŃSKI on June 20, 1959

Introduction

Bromoform, like chloroform, forms an intermolecular hydrogen bond with organic bases. In a previous work [1] the author showed that the extreme intensification (Fig. 1) of the infrared absorption band, corresponding to the C-H bond stretching vibration of bromoform in solution, is a result of the formation of a complex by means of a hydrogen bridge. On the other hand, the crioscopic data [2] provide conclusive evidence of the existence of more complicated compounds in the bromoform pyridine system, the structure of which may be represented by the formula: $3C_5H_5N.CHBr_3$. In this paper we investigate the Raman lines corresponding to the vibrations of the CBr_3 group of bromoform solved in diethylamine. From changes in the Raman spectra of this system, we assume that the existence of an intermolecular bond between the bromine atom of the bromoform and the nitrogen atom of the solvent is not excluded. It may be suggested that the nitrogen atom of diethylamine molecules interacts not only by forming a hydrogen bond. A direct action of the nitrogen atom on bromine atoms may also be taken into account. The results of the present work do not contradict the previous work [1] of the author, but supplement it.

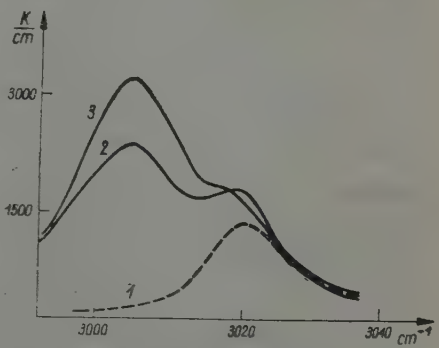

Fig. 1. C-H stretch of bromoform in solution: 1. pure $CHBr_3$; 2. 60% $CHBr_3$; 3. 40% $CHBr_3$ in $(C_2H_5)_3O$

Experimental

The Raman spectra of bromoform solutions in diethylamine of the following compositions: 66, 50, 33, 25, 20, 17 per cent mole of $CHBr_3$ were investigated.

The measurements were made at two different temperatures $+10°C$ and $-60°C$. A description of the apparatus used is given in [3]. The intensity distribution of the lines was obtained by the usual photometrical method. The relative intensities were determined by measurements of the areas under the curves of the intensity distribution by means of a planimeter. In the relative intensity measurements, the intensity of the line $\nu_3 = 227\ cm^{-1}$ was taken as standard. The overlapping lines ν_2 and ν_5 were resolved by assuming that these Raman bands have a symmetrical contour.

Results and discussion

The data in the Table show the influence of the solvent on the frequencies ν, the half-widths σ and intensities I of the Raman lines ν_2, ν_3, ν_6. The line $\nu_2(A) = 539\ cm^{-1}$ belongs to the frequency of the fully symmetric C-Br stretching vibration and the lines $\nu_3(A) = 227\ cm^{-1}$ and $\nu_6(E) = 154\ cm^{-1}$ to the symmetric and antisymmetric C-Br bending vibrations, respectively. The increase in the width of lines belonging to the bending vibration is quite considerable (about 30-40%). A narrowing of these lines probably results from quite considerable interaction between the bromine atoms and the solvent. The frequency and the intensity of these lines, however, do not change. The C-Br stretching vibrations are considerably affected. Fig. 2 shows the pattern of the variations in the shape of the ν_2 line as a function of the solution composition at a temperature of $-60°$ C. At this temperature the changes are most prominent. It would appear that the contour of the ν_2 line is the result of the superposition of three lines. The appearance of at least one new line $\nu_2^* = 533\ cm^{-1}$ belonging to the associated molecules, can be stated with certainty. Its intensity increases as the concentration of the $CHBr_3$ decreases, while the line $\nu_2 = 539\ cm^{-1}$ gradually vanishes. A very distinct change of shape is observed for the degenerate line $\nu_5\ (E) = 655\ cm^{-1}$ (Fig. 3). It seems that the contour of this line at con-

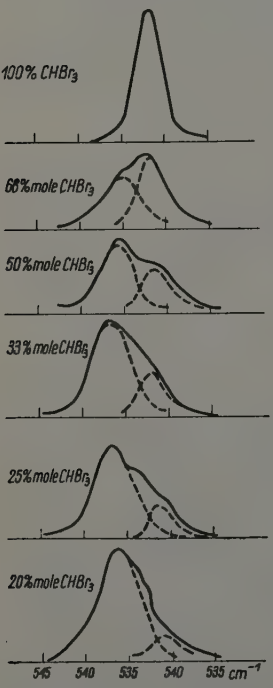

Fig. 2. Variation of contour of the ν_2 and ν_2^0 lines of $CHBr_3$ with solution composition at $-60°C$

centrations of 50 and 33% mole $CHBr_3$ results from the superposition of three lines. As a result of the solute-solvent interaction, there is a distinct superposition of two lines belonging to the perturbed molecules on a wide band (half-width 20 cm^{-1}) belonging to the unperturbed ones. At a temperature of $-60°$ C and concentrations of 20 and 17 per cent mole CHBr the doubly degenerate C-Br stretching mode $\nu_5(E)$ is split into two components of frequencies 653 cm^{-1} and 641 cm^{-1}. We assume that degeneracy is removed. The half-width

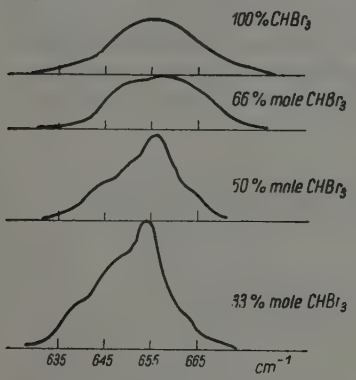

Fig. 3. Variation of contour of the ν_5 of $CHBr_3$ with solution composition at 10°C

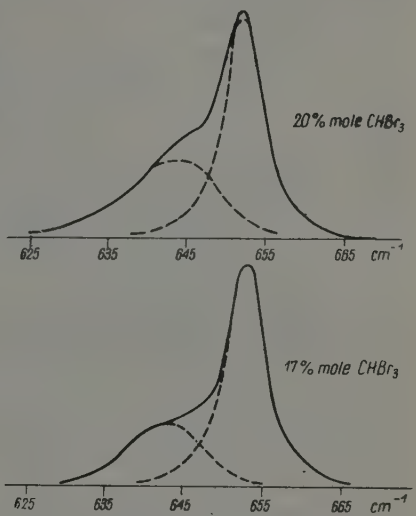

Fig. 4. Variation of contour of the ν_5 line of $CHBr_3$ in solution at $-60°$C

of the more intense component amounts to 7 cm^{-1} only, while the width of the weaker one is 11 cm^{-1}. It is interesting to note that a line correspon-

TABLE

Raman lines of the CBr_3 group vibrations of bromoform in solutions with diethylamine at 10° C.

% mole of $CHBr_3$	ν_2			ν_2^*			ν_3			ν_6		
	ν cm^{-1}	σ cm^{-1}	I	ν cm^{-1}	σ cm^{-1}	I	ν cm^{-1}	σ cm^{-1}	I	ν cm^{-1}	σ cm^{-1}	I
100	539	3.8±0.2	45				227	3.5±0.2	100	154	4.6±0.2	78
66	539	3.8±0.2	27	535	4.2±0.2	16	227	3.8±0.2	100	154	4.6±0.2	72
50	539	4±0.3	17	534.6	5±0.2	28	227	4.6±0.2	100	154	4.6±0.2	80
33	~539	~4	~9	534	5±0.2	34	227	4.6±0.2	100	154	5.2±0.2	71
25	~539	~3.6	~8	533	5.2±0.2	36	227	4.6±0.3	100	154	5.6±0.1	70
20	~539	~3.8	~4	533	5.3±0.1	40	227	5.1±0.3	100	154	6.4±0.2	78
17				533	6±0.1	40						

ν, σ, I – the frequency, half-width and intensity of Raman line.

ding to the same vibration in chloroform investigated under the same conditions [3] showed no significant narrowing and displacement. In the complex $Br_3CH - NH(C_2H_5)_2$ the electron density in the carbon atom of $CHBr_3$ is augmented by the influence of hydrogen bonding and this redistribution of the electronic charge causes a variation in the electronegativity of the bromine atoms. Such a decrease in the electronegativity of the bromine atoms can contribute to the creation of direct loose bonding of the charge transfer type [4] between the nitrogen atom of the amine and the bromine atom of the bromoform. It would thus seem reasonable to assume that, besides the complexes associated with a hydrogen bond, there exist in the bromoform diethylamine solution more complicated complexes consisting of at least two diethylamine molecules to every $CHBr_3$ molecule. Such a hypothesis is supported by our spectral data and is also in agreement with crioscopic data [2]. The absence of analogous compounds in chloroform solution can be attributed to the greater electronegativity of the chlorine atom.

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EXPERIMENTAL PHYSICS

Internal Conversion Electrons of ^{167}Tu

by

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Presented by A. SOŁTAN on June 22, 1959

Investigations on radiation of ^{167}Tu were carried out several times ([1], [2]). A scintillation spectrometer was used for the observation of gamma transitions with energies: 720, 515, 202, 115 and 49 keV [3]. The existence of 207.5 and 56.7 keV transitions has been established by the measurement of the spectrum of internal conversion electrons [4]. The same paper gives the upper limits of the intensity of conversion lines with energies of 720, 515, 115 keV as: 0.03%, 0.03%, 0.1% of the intensity of the L conversion line of the 207.5 keV transition respectively. The aim of our measurements was to find conversion lines of these three transitions as well as 350 and 370 keV transitions which could be expected on the basis of the suggested level scheme of ^{167}Er [1]. For this purpose a thick lens beta-spectrometer of high luminosity was used [5].

^{167}Tu isotope was obtained in the reaction of spallation of tantalum nuclei bombared with 660 MeV protons. The tantalum target weighing 5g was irradiated in the internal beam of the synchrocyclotron of the Joint Institute for Nuclear Research at Dubna. The reaction products were separated according to atomic number by the chromatographic method. First of all ^{165}Tu ($T_{1/2} = 29$h) and ^{167}Tu ($T_{1/2} = 9.6$ days) were expected in the thulium fraction. The measurements of the conversion spectrum of ^{167}Tu were started 14 days after the irradiation of the target and after the thulium fraction had been separated from it. The intensity of the radiation of ^{165}Tu was negligible.

For the purpose of preparing a source for the spectrometer a few drops of a solution of thulium chloride were dried on 1.7 mg/cm^2 thick aluminium foil. Next, on the basis of the obtained radio-autograph, a disc of 4 mm in diameter was cut out of the surface with the maximum activity. This disc was glued onto an aluminium foil of the same thickness.

A measurement of the electron spectrum in the momentum interval $H_\varrho = 610$ to 4170 gauss.cm was made. Observations gave strong conversion lines of transitions: 207,5 keV and 56.7 keV as well as the line of Auger electrons K-L-L with energy of about 39 keV. Besides, there were found two much weaker conversion lines with energy of electrons of 477 keV and 70.8 keV. To identify these lines additional measurements were made 8 days later. Due to the decrease of intensity with time it was established that the 477 keV line should be attributed to the isotope ^{167}Tu, and that the 70.8 keV line belongs to an isotope of a longer half life and that it is probably the conversion line L of 80 keV transition from the decay of ^{168}Tu ($T_{1/2} = 85$ days). The remaining expected lines were not observed and only the upper limits of their intensity could be estimated.

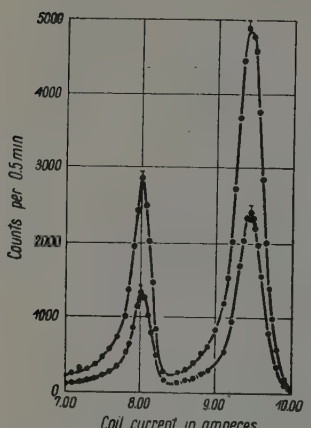

Fig. 1. K and $L+M+N$ conversion lines of the 207.5 keV transition. The measurements were made on the 14th and 24th day after the target bombardment

Fig. 1 shows lines K and $L+M$ of the 207.5 keV transition. As the energy of these lines was measured rather accurately [4], the line K was used for the calibration of the spectrometer. Its intensity was taken to be 100%. Fig. 2 shows the 477 keV line.

The final results of the measurements are given in Table I.

TABLE I

No.	Energy of gamma transition according to [1]	Energy of conversion electr.	Intensity in relation to $K-207.5$	Interpretation	Energy of gamma transition
1	115 keV	—	<0.3%	—	—
2	350	—	<0.04	—	—
3	370	—	<0.04	—	—
4	515	476.9 ± 7.7 keV	0.11 ± 0.04	conversion in K-shell	534 ± 8 keV
5	720	—	<0.03	—	—

Note: For transitions 2, 3 and 5 the estimation of the upper limit of intensity refers to lines K and L. For transition 1 only the line L was taken into consideration because the line K is masqued by the line M of the 56.7 keV transition. In the case of transition 4 the relation of intensities was estimated to be $\frac{K}{L} > 2.8$.

According to [6] the internal conversion coefficient in the K – shell for the 207.5 keV transition amounts to $\alpha_k = 0.55 \pm 0.10$. According to [3] the relation of the intensities of gamma lines 207.5 and 534 keV amounts to 0.29/0.09. Using the above data and the results obtained in our work in measurements of the relations of intensities of internal conversion electron lines it is possible to calculate the value of the con-

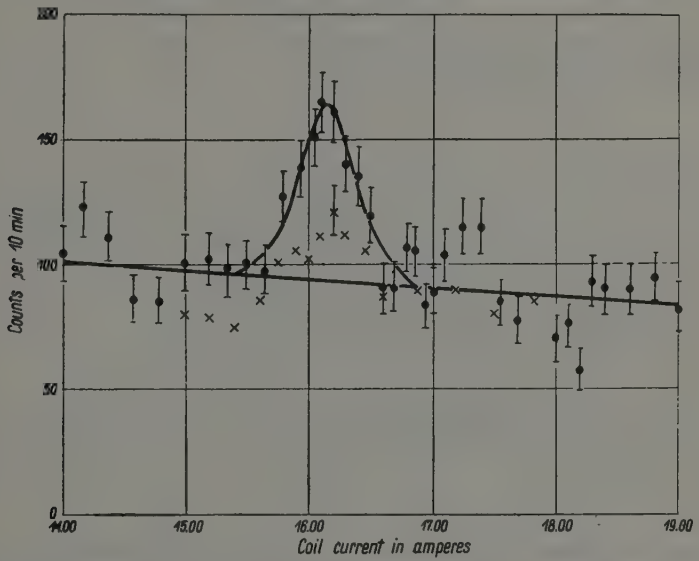

Fig. 2. K conversion line of the 534 keV transition. The measurements were made 15 days after the target bombardment and repeated 8 days later (crosses)

version coefficient of K shell for the 534 keV transition. It amounts to: $\alpha_k = (2.0 \pm 0.8) \times 10^{-3}$. The accuracy in establishing the relation of the intensities of gamma lines was not taken into consideration when estimating the limits of error.

Table II gives theoretical internal conversion coefficients of K-shell for transition of 534 keV energy according to Rose's Tables [7]. According to this Table, the transition under discussion is of the type $E1$ (electrical dipole). Thus the corresponding spin of the 742 keV level (207.5 keV + 534 keV, see [4] or [1]) should be 1/2 or 3/2 and the partity even.

TABLE II

Transition	$E1$	$E2$	$E3$	$M1$	$M2$	$M3$
Conversion coefficient of K-shell	4.0×10^{-3}	1.1×10^{-2}	2.7×10^{-2}	2.5×10^{-2}	7.5×10^{-2}	1.8×10^{-1}

It is necessary, however, to take into consideration the possibility of a change of the interpretation that 534 keV transition is $E1$, for, in [8], a corresponding gamma line was not observed. However, it was stated in [8] that, should the gamma transition take place, then this probability would be at least 1 order of magnitude smaller than given in [3]. In this case the conversion coefficient would be larger than 2.0×10^{-3}.

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БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

Резюме статей

ТОМ VII . . . . . 1959 ВЫПУСК 8

Классические теоремы Герглотца и Бохнера были обобщены А. Вейлем, Д. Райковым и автором настоящей работы. Кроме того, в последнее время были приведены разложения ядер по собственным дистрибуциям волновых операторов (теорема Челлена-Леманна).

В настоящей работе показано, каким образом эти теоремы могут быть обобщены путем метода спектрального представления довольно общих ядер собственными ядрами перестановочных операторов в некотором соответственно подобранном пространстве Гильберта.

Получается также интересный вывод, а именно новое доказательство теоремы Понтрягина, говорящей о том, что группа характеров дискретной группы является компактной.

1. Пусть Φ плотное линейное подмножество сепарабельного пространства Гильберта H такое, что вложение $\Phi \to H$ является ядерным (напр., если Φ — ядерное); тогда для коммутативной переменной системы самосопряженных операторов в H (непрерывных на Φ) имеет место теорема о разложении по обобщенным собственным функциям типа Гельфанда—Костюченки.

2. Приводится конструкция пространства Φ для счетной системы операторов в H.

3. Для абстрактных ядер, т. е. элементов пространства $\Phi' \otimes \Phi'$ имеет место обобщение доказанной автором в предыдущей работе [9] теоремы о спектральном представлении ядер.

4. Пусть $G \ni g \to T(g) \in \mathcal{L}H)$ — унитарное представление (сепарабельной) группы Ли в пространстве Гильберта H. Доказывается, что каждое такое представление индуцирует ядерное пространство Φ, а также так называемые

„T-распределения". Если $g \to L(g)$ — регулярное представление, то L — распределения являются (обыкновенными) распределениями Шварца. Эта конструкция позволяет перенести теоремы о сферических распределениях на группах Ли, приведенных автором в работе [8], на произвольные представления групп Ли.

П. ТУРАН, ЗАМЕТКА О ПРЕДСТАВЛЕНИИ В ФОРМЕ БЕСКОНЕЧНОГО ПРОИЗВЕДЕНИЯ ФУНКЦИЙ РЕГУЛЯРНЫХ И ОТЛИЧНЫХ ОТ НУЛЯ В ЕДИНИЧНОМ КРУГЕ . . . . . стр. 481—486

Пусть C обозначает класс функций $F(x)$ комплексного переменного, которые регулярны в области $|z| < 1$ и непрерывны для $|z| \leqslant 1 \mu F(o) = 1$. В работе содержится доказательство следующей теоремы:

Теорема. *Если $F(z) \in C$, то существуют действительные постоянные d_n такие, что ряд*

$$\sum_{n=1}^{\infty} d_n \log f_n(z)$$

сходится к $F(z)$ для $|z| < 1$, причем для каждого $\varepsilon > 0$ сходимость его в круге $|z| \leqslant 1 - \varepsilon$ является равномерной. Кроме того, ряд

$$\sum_{n=1}^{\infty} d_n \operatorname{Im} \log f_n(z)$$

является для $|z| \leqslant 1$ равномерно сходящимся к $\operatorname{Im} F(z)$.

Ч. ОЛЕХ, ОЦЕНКИ ЭКСПОНЕНЦИОНАЛЬНОГО РОСТА РЕШЕНИЙ ОБЫКНОВЕННОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ВТОРОГО ПОРЯДКА . . . . . стр. 487—494

Рассмотрим уравнение (1) $x'' + 2a(t)x' + b(t)x = 0$, с коэффициентами $a(t)$ и $b(t)$; предположим, что они кусочнонепрерывны и ограничены. Применяя метод из своей предыдущей работы [1], автор определяет две постоянные ϱ_1 и ϱ_2 таким образом, что для произвольного, нетривиального решения $x(t)$ уравнения (1) и произвольного $\varepsilon > 0$ выполняются следующие неравенства

$$(2) \quad \limsup_{t \to +\infty} |x(t)| \exp\left(-(\varrho_1 - \varepsilon)t\right) > 0, \quad \limsup_{t \to +\infty} |x(t)| \exp\left(-(\varrho_2 + \varepsilon)t\right) < +\infty.$$

Величины ϱ_1 и ϱ_2 — однозначно определенные постоянными ограничивающими коэффициенты уравнения (1), а при заданных постоянных ограничивающих можно так определить функции $a(t)$ и $b(t)$, что для некоторых решений уравнения (1) первое (либо второе) из соотношений (2) не выполняется для $\varepsilon < 0$.

Автор строит эффективно уравнения, которые выполняются постоянными ϱ_1 и ϱ_2.

Автор рассматривает дифференциальное уравнение

$$x''+F(x')+g(x)=p(t) \tag{1}$$

при предположении, что функции $g(x)$, $F(y)$ и $p(t)$ регулярны соответственно и удовлетворяют условиям

$$\lim_{|x|\to\infty} g(x)\operatorname{sgn}x=+\infty\,,\quad \lim_{|y|\to\infty} F(y)\operatorname{sgn}y=+\infty\,,\quad |p(t)|\leqslant P\,.$$

На основании теоремы Г. Рейтера [6] известно, что существуют постоянные B_1 и B_2 такие, что для каждого решения $x(t)$ уравнения (1) для достаточно больших t имеют место неравенства $|x(t)|\leqslant B_1$, $|x'(t)|\leqslant B_2$. Автор показывает, что при дополнительных предположениях

$$f(y)=F'(y)>0\,,\ g'(x)>0\,,$$
$$2\min f(y)/|y|>\max|g''(x)|/g'(x)\ (|x|\leqslant B_1\,,\ |y|\leqslant B_2)$$

для произвольных двух решений $x_1(t)$, $x_2(t)$ уравнения (1) имеем

$$\lim_{t\to+\infty}|x_1(t)-x_2(t)|=\lim_{t\to+\infty}|x_1'(t)-x_2'(t)|=0\,.$$

Если функция $p(t)$ является периодической либо почти периодической, то отсюда вытекает существование единственного периодического либо почти периодического решения уравнения (1).

Терминология в настоящей работе согласуется с [2].

Если $n\geqslant 4$ и алгебра независимо генерирована через каждое n элементов (т. е., каждое множество, состоящее из n элементов алгебры, является множеством независимых генераторов этой алгебры), тогда в этой алгебре единственно функции формы $(*)$ являются алгебраическими функциями (Теорема 2).

В работе приводится также дискуссия случаев $n=1,2,3$.

Доказательства теорем будут опубликованы в *Fundamenta Mathematicae*.

Рассматривается плоский случай магнитогравитационной неустойчивости невязкой, изотермической, неоднородно вращающейся среды с большой электропроводностью, находящейся под действием возмущения с осевой симметрией. Магнитное поле среды предполагается в форме круговых силовых линий сим-

метрических относительно оси вращения системы. Получена следующая форма условия неустойчивости:

$$-4l^2(V_a^2+V_s^2)^2-r_0^{-2}\{V_s^4+9V_a^4+6V_a^2V_s^2\}+$$
$$+4(V_a^2+V_s^2)\{2\Omega F+4\pi G\varrho+4V_a^2r_0^{-2}\}>0\,,$$

где r_0 — расстояние от оси вращения системы, V_s — скорость звука, ϱ — плотность среды, Ω — угловая скорость вращения,

$$F=-\frac{d}{dr}(\Omega r)-\Omega$$

и

$$V_a^2=H_\varphi^2(4\pi\varrho)^{-1}$$

(H_φ — напряженность магнитного поля). l и λ обозначают: волновое число и длину волны соответственно. При большом r_0 это условие принимает более простой вид:

$$\lambda>\lambda_*=\pi\sqrt{\frac{V_s^2+V_a^2}{\pi G\varrho+\frac{1}{2}\Omega F}}.$$

Приведенное условие определяет максимальную величину напряженности магнитного поля, которая могла существовать в газовой протогалактике, в которой возникли спиральные ветви вследствие действия вышерассмотренного механизма неустойчивости. Эта максимальная напряженность магнитного поля равняется $1{,}0\cdot10^{-5}$ *гаусс* при принятых размерах спиральных ветвей $\frac{1}{2}\lambda_*=1$ *кпс*, плотности протогалактики вблизи Солнца $\varrho=4{,}2\cdot10^{-24}$ *г·см*$^{-3}$, скорости звука $V_s=2{,}3$ *км·сек*$^{-1}$ и величине фактора $\frac{1}{2}\Omega F=-0{,}37\cdot10^{-30}$ *сек*$^{-2}$.

Малые отклонения наблюдаемые в бета-спектрах [1]—[4] можно было линеаризовать единственно путем введения эмпирического члена типа Фирца с таким же знаком для электронных и позитронных спектров. В представленной работе получены члены этого типа в спектре при простом предположении полуфеноменологического рода. Ввиду того, что интерференция $A-T$ в переходах Гамова—Теллера считается теперь почти невозможной, остается объяснение экспериментального спектра путем принятия добавочных матричных элементов, которые могут преобладать над обычно выступающими элементами.

Представленная схема опирается на идеях Гелль-Манна.

Вычислены разницы масс между барионами в теории тяжелых частиц, данной В. Круликовским. Получено $\Delta m=160\,m_e$ между частицами Σ и Λ, $750\,m_e$ для Ξ и N, $317\,m_e$ для $\Lambda-N$, в хорошем согласии с опытом.

Вычисление сделано в не релятивистской теории возмущений при $\frac{f_0^2\pi}{4\pi} = 0{,}08$ и $\frac{f_0^2 K}{4\pi} = 0{,}25$. Сечение сделано: $k_{\max} = 3{,}5\,\mu_\pi$ для π и $1{,}4\,\mu K$ для K — взаимодействия.

В связи с возможностью виртуального перехода $\Lambda^0 \leftrightarrow \Sigma$ в гиперфрагментах, в работе изучается π^+ распад гиперфрагмента ${}_\Lambda He^4$(${}_\Lambda He^4 \to H^3 + n + \pi^+$). Процесс изучается в найнизшем одномезонном приближении. Оценивается отношение вероятности распада ${}_\Lambda He^4$ на π^+ к вероятности распада на π^-. Качественно оценивается вклад от взаимодействия через K-мезоны. По порядку величины, для исследования отношения получаем $1 \cdot 10^{-2}$ [020].

Были проведены измерения пропускания нейтронов с использованием непрерывного спектра нейтронов из реактора (в области надкадмиевых нейтронов с борным детектором). Кривые зависимости $\ln T$ от толщины образца получены для Al, Ag и Bi. Наклон кривых при больших толщинах образца кажется справедливым толковать как эффективное сечение потенциального рассеяния.

Исследованы линии спектра Рамана, отвечающие частотам колебаний группы CBr_3 бромоформа в растворах с диэтиламином при температурах $-60°$ и $+10°$ С. Линии, соответствующие частотам деформационных колебаний $\nu_3(A_1) = 227\ см^{-1}$ и $\nu_6(E) = 154\ см^{-1}$ расширяются в значительной степени (около 30-40%).

Наблюдаем новую линию $\nu_2^* = 533\ см^{-1}$, сдвинутую на $6\ см^{-1}$ в сторону малых частот по отношению к частоте полносимметричного валентного колебания $\nu_2(A)$. Кроме того, наблюдаем снятие дегенерации для валентного, несимметричного, дважды вырожденного колебания $\nu_5(E) = 655\ см^{-1}$. Длина отвечающая этому колебанию разлагается на две составляющие, частоты которых равняются $653\ см^{-1}$ и $641\ см^{-1}$, а полуширины их равняются $7\ см^{-1}$ и $11\ см^{-1}$ соответственно.

Предполагается, что между азотом амина и бромом бромоформа образуется слабая междумолекулярнуя связь.

Исследован спектр электронов внутренней конверсии ^{167}Tu. Найденная новая конверсионная линия 477 *кэв*, которая истолкована как K — линия гамма перехода 534 *кэв*.

На основании измерений, проведенных в настоящей работе и данных (в правильности которых авторы, однако, не вполне уверены), опубликованных другими авторами — можно приписать переходу 534 *кэв* мультиполность $E1$.